

# A representation theorem on a filtering model with first-passage-type stopping time

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## Abstract

We present a representation theorem for a filtering model with first-passage-type stopping time. The model is constructed from two unobservable processes and one observable process that is under the influence of two unobservable processes. A filter is constructed using Brownian motion in the observable process and a first-passage-type stopping time in an unobservable process. Though our theorems are similar to those of Nakagawa[5], we do not use pinned Brownian motion measure, which is difficult to deal with. In addition, we describe a representation theorem for another filtration that was not discussed by Nakagawa[5].

## 1 Introduction

Duffie and Lando [2] studied the implications of imperfect information for the term structures of credit spreads on corporate bonds. They assumed that the bond investor could not observe the issuer's assets directly, and could receive only periodic and imperfect accounting information. They then derived a relationship between the volatility of the issuer's asset value and its hazard rate. Their model is a kind of filtering model. Jeanblanc and Valchev [4] examined three types of information related to a company's unlevered asset value on the secondary bond market: the classical case of continuous and perfect information, observations of past and contemporaneous asset values at selected discrete times, and observations of contemporaneous asset values at discrete times. In their model, although bond holders receive information about contemporaneous and past asset values in the second type of information, they receive only contemporaneous information in the third type. Jarrow, Protter

and Deniz [3] provided an alternative credit risk model based on information reduction, whereby the market only observes the company's asset value when it reaches certain levels, interpreted as changes significant enough for the company's management to make a public announcement. Nakagawa[5] constructed a filtering model based on a default risk, and derived representation formulas under conditions of imperfect information. He analyzed the properties of processes under  $\nu_{0,x_1}^{u,x_2}$ , which is a probability measure on  $C([0, u]; \mathbf{R})$ , and the law of Brownian motion  $B_t$  conditioned to start from  $x_1 > 0$ , stay in  $(0, \infty)$  for  $s \leq u$  and reach  $x_2 > 0$  at time  $u$  under  $P$ . However, because this measure is difficult to deal with, we present representation formulas that do not use the measure  $\nu$ . In this paper, we refer to the “first-passage-type stopping time” instead of a “default time”, because our focus is solely on the mathematical perspective of a filtering model.

First, we present a representation theorem for a filtration with first-passage-type stopping time. In this part, we do not use a filtration model.

Let  $(\Omega, \mathcal{B}, P, \{\mathcal{B}_t\}_{t \geq 0})$  be a complete filtrated probability space, and assume that the filtration  $\{\mathcal{B}_t\}_{t \geq 0}$  satisfies the usual conditions. Let  $B_t$ ,  $\hat{B}_t$  and  $W_t$  be independent  $\mathcal{B}_t$ -Brownian motions with values in  $\mathbf{R}, \mathbf{R}^d$  and  $\mathbf{R}$  respectively. We denote the right continuous filtration generated by a continuous stochastic process  $X$  as  $(\mathcal{G}_t^X)$ . For example,  $\mathcal{G}_t^B = \bigcap_{u > t} \sigma\{B_s, s \leq u\}$ . Let  $a > 0$ ,  $B_t^a = a + B_t$ ,  $\tau^a = \inf\{t > 0; B_t^a = 0\}$ ,  $N_t^a = 1_{\{\tau^a \leq t\}}$  and  $\mathcal{F}_t^W = \bigcap_{u > t} (\mathcal{G}_u^W \vee \sigma\{\tau^a \wedge u\})$ . Let  $q_a(t) = \int_t^\infty \frac{a}{\sqrt{2\pi s^3}} \exp(-\frac{a^2}{2s}) ds$  and  $\lambda_a(t) = -\frac{d}{dt} \log q_a(t) = \frac{a}{\sqrt{2\pi t^3}} q_a(t)^{-1} \exp(-\frac{a^2}{2t})$ . Then  $P[\tau^a > t] = q_a(t) = e^{-\int_0^t \lambda_a(u) du}$ . Let  $\gamma_a(t)$  be the density of  $\tau^a$ . Then, we have

$$\gamma_a(t) dt = P[\tau^a \in dt] = \lambda_a(t) e^{-\int_0^t \lambda_a(u) du} dt. \quad (1)$$

We can also see that

$$M_t^a = N_t^a - \int_0^t (1 - N_s^a) \lambda_a(s) ds$$

is  $\mathcal{F}_t^W$ -martingale.

Let  $g(t, x)$  and  $\Phi(t, x)$  be the density and distribution, respectively, of the Brownian motion  $B_t$ . Hence,  $g(t, x)$  and  $\Phi(t, x)$  can be written as follows.

$$g(t, x) = \frac{1}{\sqrt{2\pi t}} \exp(-\frac{x^2}{2t}), \quad \Phi(t, x) = \int_{-\infty}^x g(t, y) dy, \quad x \geq 0, t > 0. \quad (2)$$

We note that

$$\frac{\partial g}{\partial x}(t, x) = -\frac{x}{t} g(t, x), \quad \frac{\partial^2 g}{\partial x^2}(t, x) = 2 \frac{\partial g}{\partial t}(t, x) = \frac{x^2 - t}{t^2} g(t, x).$$

We denote as  $\mathcal{L}^p$ ,  $p \in (1, \infty)$ , the space of  $\{\mathcal{B}_t\}$ -progressively measurable functions  $\varphi$  such that  $E[\int_0^T |\varphi|_s^p ds] < \infty$  for any  $T > 0$ , and write  $\mathcal{L}^{p+} = \bigcup_{q>p} \mathcal{L}^q$ ,  $p \geq 1$ . For  $t > s$ , let

$$H_a^{(k)}(t, s; f) = E[1_{\{\tau^a > s\}} f_s \frac{\partial^k g}{\partial x^k}(t - s, B_s^a) | \mathcal{G}_s^W], \quad f \in \mathcal{L}^{1+}, \quad (3)$$

$$k = 0, 1, 2,$$

$$\hat{H}_a^{(k)}(t; f) = \int_0^t H_a^{(k)}(t, u; f) du, \quad f \in \mathcal{L}^{\frac{4}{3-k}+}, \quad k = 0, 1, 2, \quad (4)$$

$$\bar{H}_a(t; f) = e^{\int_0^t \lambda_a(r) dr} \{ \hat{H}_a^{(2)}(t; f) + 2\lambda_a(t) \hat{H}_a^{(0)}(t; f) \}, \quad (5)$$

$$f \in \mathcal{L}^{4+},$$

$$U_a(t, s; f) = E[1_{\{\tau^a > s\}} f_s (2\Phi(t - s, B_s^a) - 1) | \mathcal{G}_s^W], \quad f \in \mathcal{L}^{1+} \quad (6)$$

$$\bar{U}_a(t, s; f) = e^{\int_0^t \lambda_a(r) dr} \{ H_a^{(1)}(t, s; f) + \lambda_a(t) U_a(t, s; f) \}, \quad (7)$$

$$f \in \mathcal{L}^{1+}.$$

We will show that these are well defined in Section 2. Thus we have the following theorem.

**Theorem 1.1** (1) For any  $t, T \geq 0$  and  $f \in \mathcal{L}^{4+}$ ,

$$E\left[\int_0^T f_s dB_s | \mathcal{F}_t^W\right] = - \int_0^t \bar{H}_a(s; f 1_{(0,T]}(\cdot)) \lambda_a(s)^{-1} dM_s^a.$$

(2) For any  $t \geq 0$  and  $f \in \mathcal{L}^{4+}$ ,

$$E\left[\int_0^t f_s ds | \mathcal{F}_t^W\right] = \int_0^t E[f_s | \mathcal{F}_s^W] ds - \int_0^t \left( \int_0^s \bar{U}_a(s, r; f) dr \right) \lambda_a(s)^{-1} dM_s^a.$$

(3) For any  $t, T \geq 0$  and  $f \in \mathcal{L}^{6+}$ ,

$$E\left[\int_0^T f_s dW_s | \mathcal{F}_t^W\right] = \int_0^{T \wedge t} E[f_s | \mathcal{F}_s^W] dW_s - \int_0^t \left( \int_0^s \bar{U}_a(s, r; f 1_{[0,T]}(\cdot)) dW_r \right) \lambda_a(s)^{-1} dM_s^a.$$

(4) For any  $t \geq 0, \hat{f}_i \in \mathcal{L}^{2+}, i = 1, \dots, d$ ,

$$E[\hat{f}_s^i d\hat{B}_s^i | \mathcal{F}_t^W] = 0.$$

Second, we consider a representation theorem with a filtering model. The quantities  $X$ ,  $Z$ , and  $Y$  are the same as those considered by Nakagawa[5], and are called the main system, sub-system and observation, respectively, in

his paper. Let  $X$  and  $Z$  be solutions of the following stochastic differential equations under  $P$ :

$$\begin{aligned} dX_t &= dB_t + b_0(t, X_t, Z_t)dt, & X_0 &= x_0 > 0, \\ dZ_t &= \sigma_1(t, X_t, Z_t)d\hat{B}_t + b_1(t, X_t, Z_t)dt, & Z_0 &= z_0 \in \mathbf{R}^N, \end{aligned}$$

where  $b_0 : [0, \infty) \times \mathbf{R} \times \mathbf{R}^N \rightarrow \mathbf{R}$ ,  $\sigma_1 : [0, \infty) \times \mathbf{R} \times \mathbf{R}^N \rightarrow \mathbf{R}^{N \times d}$  and  $b_1 : [0, \infty) \times \mathbf{R} \times \mathbf{R}^N \rightarrow \mathbf{R}^N$  are bounded and continuously differentiable functions. Let  $Y$  be a solution of the stochastic differential equation,

$$dY_t = \sigma_2(t, Y_t)dW_t + b_2(t, X_{t \wedge \tau}, Y_t)dt, \quad Y_0 = y_0 \in \mathbf{R},$$

where  $\sigma_2 : [0, \infty) \times \mathbf{R} \rightarrow \mathbf{R}$  and  $b_2 : [0, \infty) \times \mathbf{R} \times \mathbf{R} \rightarrow \mathbf{R}$  are bounded and continuously differentiable functions. We assume that there exist some  $\epsilon > 0$  and  $\sigma_2(t, y)$  satisfying  $\sigma_2(t, y) \geq \epsilon$  for any  $t \in [0, \infty)$ ,  $y \in \mathbf{R}$ . Let  $\tau = \inf\{t > 0; X_t = 0\}$ ,  $N_t = 1_{\{\tau \leq t\}}$  and  $\mathcal{F}_t = \bigcap_{u>t} (\mathcal{G}_u^Y \vee \sigma\{\tau \wedge u\})$ . We now consider changing the probability measure. Let  $\rho_t$  be given by

$$\begin{aligned} \rho_t &= \exp \left( \int_0^t b_0(s, X_s, Z_s)dB_s + \int_0^t \beta(s, X_{s \wedge \tau}, Y_s)dW_s \right. \\ &\quad \left. + \frac{1}{2} \int_0^t (b_0(s, X_s, Z_s)^2 + \beta(s, X_{s \wedge \tau}, Y_s)^2)ds \right), \end{aligned} \quad (8)$$

where  $\beta(t, x, y) = \sigma_2(t, y)^{-1}b_2(t, x, y)$  and  $\tilde{P}$  is a probability measure on  $(\Omega, \mathcal{F})$  given by  $d\tilde{P} = \rho_t^{-1}dP$ . We can see that  $\rho, \rho^{-1} \in \bigcap_{p \geq 1} \mathcal{L}^p$  by Novikov's Theorem. Let  $\tilde{\rho}_t = \tilde{E}[\rho_t | \mathcal{F}_t]$ . Here, we will denote the expectation under the probability measure  $\tilde{P}$  as  $\tilde{E}[\cdot]$ . Let

$$\begin{aligned} \tilde{B}_t &= B_t + \int_0^t b_0(s, X_s, Z_s)ds, \\ \tilde{W}_t &= W_t + \int_0^t \beta(s, X_{s \wedge \tau}, Y_s)ds. \end{aligned}$$

Then  $\tilde{B}_t$ ,  $\hat{B}_t$  and  $\tilde{W}_t$  are independent  $\tilde{P}$ - $\{\mathcal{B}_t\}_{t \in [0, \infty)}$ -Brownian motions. The stochastic processes  $X$ ,  $Z$  and  $Y$  are described in the following:

$$\begin{aligned} dX_t &= d\tilde{B}_t, \\ dZ_t &= \sigma_1(t, X_t, Z_t)d\hat{B}_t + b_1(t, X_t, Z_t)dt, \\ dY_t &= \sigma_2(t, Y_t)d\tilde{W}_t. \end{aligned}$$

From the above equations, we can see that  $\{\mathcal{G}_t^X\}_{t \in [0, \infty)}$  coincides with the natural filtration generated by  $\{\tilde{B}_t\}_{t \in [0, \infty)}$ . Because  $d\tilde{W}_t = \sigma(t, Y_t)^{-1}dY_t$ , we

can see that  $\mathcal{G}_t^Y = \mathcal{G}_t^{\widetilde{W}}$  and  $\mathcal{F}_t = \bigcap_{u>t} (\mathcal{G}_u^{\widetilde{W}} \vee \sigma\{\tau \wedge u\})$ . In addition, we can see that

$$\widetilde{M}_t = N_t^{x_0} - \int_0^t (1 - N_{s-}^{x_0}) \lambda_{x_0}(s) ds$$

is  $\widetilde{P}$ - $\mathcal{F}_t$ -martingale. Let

$$I^{(k)}(t, s; f) = \widetilde{E}[1_{\{\tau>s\}} \frac{\partial^k g}{\partial x^k}(t-s, X_s) \rho_{s-} f_s | \mathcal{G}_s^Y], \quad t > s, \quad k = 0, 1, 2 \quad (9)$$

for  $f \in \mathcal{L}^{2+}$ . Let  $\Sigma$  denote the set of  $\mathcal{B}$ -adapted continuous processes  $F$  for which there exist  $f_i, i = 1, 2, 3 \in \mathcal{L}^{6+}$  and  $(f_4^j), j = 1, \dots, d \in \mathcal{L}^{6+}$  such that

$$F_t = F_0 + \int_0^t f_1(s) ds + \int_0^t f_2(s) dW_s + \int_0^t f_3(s) dB_s + \sum_{j=1}^d \int_0^t f_4^j(s) d\hat{B}_s^j. \quad (10)$$

For  $F \in \Sigma$ , let

$$\begin{aligned} (\widetilde{D}_0 F)_t &= \beta(t, X_{t \wedge \tau}, Y_t) F_{t \wedge \tau} + 1_{\{\tau>t\}} f_2(t), \\ (\widetilde{D}_1 F)_t &= b_0(t, X_t, Z_t) F_{t \wedge \tau} + 1_{\{\tau>t\}} f_3(t), \\ (\widetilde{D}_2 F)_t &= 1_{\{\tau>t\}} f_4(t), \\ (\widetilde{L} F)_t &= 1_{\{\tau>t\}} f_1(t). \end{aligned} \quad (11)$$

For  $r > s > 0$ , let

$$\begin{aligned} \hat{V}(r; F) &= \\ \widetilde{\rho}_{r-}^{-1} e^{\int_0^r \lambda_{x_0}(u) du} &\left( \hat{V}_1(r, r; F) + \lambda_{x_0}(r) (-(2\Phi(r, x_0) - 1) F_0 + \widetilde{E}[1_{\{\tau>r\}} \rho_r F_r | \mathcal{G}_r^Y]) \right), \\ \hat{V}_1(r, s; F) &= \\ \int_0^s I^{(1)}(r, u; \widetilde{D}_0 F) d\widetilde{W}_u &+ \int_0^s \left( I^{(2)}(r, u; \widetilde{D}_1 F) + I^{(1)}(r, u; \widetilde{L} F) \right) du. \end{aligned} \quad (12)$$

Let

$$\widetilde{\tilde{\lambda}}(s) = \lambda_{x_0}(s) + \hat{V}(s; 1), \quad \widetilde{\widetilde{M}}_t = N_t - \int_0^t (1 - N_s) \widetilde{\tilde{\lambda}}(s) ds$$

and

$$\widetilde{\widetilde{W}}_t = \widetilde{W}_t - \int_0^t E[\beta(r, X_{r \wedge \tau}, Y_r) | \mathcal{F}_r] dr.$$

Then, we will show that  $\widetilde{\widetilde{M}}_t$  is  $P$ - $\mathcal{F}_t$ -martingale and that  $\widetilde{\widetilde{W}}_t$  is a  $P$ - $\mathcal{F}_t$ -Brownian motion. Nakagawa [5] also gave  $\widetilde{\tilde{\lambda}}$  using the measure of a pinned Brownian motion. We can now state the following representation theorem, which was not given by Nakagawa [5].

**Theorem 1.2** Let  $F \in \Sigma$  and  $\bar{F}_t = E[F_{t \wedge \tau} | \mathcal{F}_t]$ . Then we have the following.  
(1)

$$\bar{F}_t = F_0 + \int_0^t \bar{f}_0(r; F) d\widetilde{M}_r + \int_0^t \bar{f}_1(r; F) dr + \int_0^t \bar{f}_2(r; F) d\widetilde{W}_r,$$

where

$$\begin{aligned} \bar{f}_0(r; F) &= -1_{\{\tau > r\}}(\hat{V}(r; F) + \hat{V}(r; 1)\bar{F}_{r-})\tilde{\lambda}(r)^{-1}, \\ \bar{f}_1(r; F) &= 1_{\{\tau > r\}}E[1_{\{\tau > r\}}(\tilde{L}F)_r | \mathcal{F}_r], \\ \bar{f}_2(r; F) &= E[(\tilde{D}_0 F)_r | \mathcal{F}_r] - E[\beta(r, X_r, Y_r) | \mathcal{F}_r]\bar{F}_{r-}. \end{aligned}$$

(2) Moreover, if there exist  $C > 0$  and  $\alpha \in (0, 1)$  such that  $1_{\{|X_t| \leq 1\}}1_{\{\tau > t\}}|F_t| \leq C|X_t|^\alpha$  for  $t > 0$ , we have  $\bar{f}_0(r; F) = -1_{\{\tau > r\}}\bar{F}_{r-}$ .

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## 2 Evaluation of integrands

For  $f \in \mathcal{L}^1$ ,  $t > s > 0$  and  $k = 0, 1, 2$ , let

$$\tilde{H}_a^{(k)}(t, s; f) = E[1_{\{\tau^a > s\}} | f_s \frac{\partial^k g}{\partial x^k}(t - s, B_s^a) | | \mathcal{G}_s^W]. \quad (13)$$

**Proposition 2.1** For  $q > 1$  and  $k = 0, 1, 2$ , we have

$$\tilde{H}_a^{(k)}(t, s; 1) \leq C_1^{(k)}(a) + C_2^{(k)}(q, a)(t - u)^{\frac{-kq - q + 2}{2}}$$

for any  $t > u \geq 0$  with  $t - u \leq 1$ . Here

$$\begin{aligned} C_1^{(k)}(a) &= \sup_{x \geq a/2, t > 0} \left| \frac{\partial^k g}{\partial x^k}(t, x) \right| < \infty, \\ C_2^{(k)}(q, a) &= 2a \left( \int_0^\infty y \left| \frac{\partial^k g}{\partial y^k}(1, y) \right|^q dy \right) \sup_{u > 0} \frac{g(u, \frac{a}{2})}{u} < \infty. \end{aligned}$$

*Proof.* We have

$$\frac{\partial^k g}{\partial x^k}(t, x) = \frac{\partial^k}{\partial x^k}(t^{-\frac{1}{2}}g(1, t^{-\frac{1}{2}}x)) = t^{-\frac{k+1}{2}} \frac{\partial^k g}{\partial x^k}(1, t^{-\frac{1}{2}}x).$$

Since  $\{B_t^a\}$  and  $\{W_t\}$  are independent,

$$\begin{aligned}
& E[1_{\{\tau^a > u\}} |\frac{\partial^k g}{\partial x^k}(t-u, B_u^a)|^q | \mathcal{G}_u^W] \\
&= \int_0^\infty (g(u, x-a) - g(u, x+a)) |\frac{\partial^k g}{\partial x^k}(t-u, x)|^q dx \\
&= \int_0^\infty g(u, x-a) (1 - \exp(-\frac{2ax}{u})) |\frac{\partial^k g}{\partial x^k}(t-u, x)|^q dx \\
&\leq \int_{a/2}^\infty g(u, x-a) |\frac{\partial^k g}{\partial x^k}(t-u, x)|^q dx \\
&+ \frac{2a}{u} \int_0^{a/2} g(u, x-a) x |\frac{\partial^k g}{\partial x^k}(t-u, x)|^q dx. \tag{14}
\end{aligned}$$

For the first term, we have

$$\int_{a/2}^\infty g(u, x-a) |\frac{\partial^k g}{\partial x^k}(t-u, x)|^q dx \leq C_1^{(k)}(a) \int_{a/2}^\infty g(u, x-a) dx \leq C_1^{(k)}(a).$$

For the second term, we have

$$\begin{aligned}
& \frac{2a}{u} \int_0^{a/2} g(u, x-a) x |\frac{\partial^k g}{\partial x^k}(t-u, x)|^q dx \\
&\leq g(u, \frac{a}{2}) \frac{2a}{u} \int_0^\infty x |\frac{\partial^k g}{\partial x^k}(t-u, x)|^q dx \\
&= g(u, \frac{a}{2}) \frac{2a}{u} \int_0^\infty x |(t-u)^{-\frac{k+1}{2}} \frac{\partial^k g}{\partial x^k}(1, (t-u)^{-\frac{1}{2}} x)|^q dx \\
&= g(u, \frac{a}{2}) \frac{2a}{u} (\int_0^\infty y |\frac{\partial^k g}{\partial y^k}(1, y)|^q dy) (t-u)^{\frac{-kq-q+2}{2}} \\
&\leq C_2^{(k)}(q, a) (t-u)^{\frac{-kq-q+2}{2}}.
\end{aligned}$$

Then we have our assertion. ■

To represent the conditional expectation under  $P$  with respect to  $\{\mathcal{G}_t^W\}$  and  $\{\mathcal{F}_t^W\}$ , we must derive some inequalities to define stochastic integrals. Propositions 2.2 and 2.3 enable us to evaluate  $\bar{H}_a$  and  $\bar{U}_a$  in Theorem 1.1. These quantities are defined in Equations (5) and (7), respectively.

**Proposition 2.2** *Let  $p \in (1, \infty)$  and  $q = \frac{p}{p-1}$ .*

(1) *For  $k = 0, 1, 2$ , there are some  $C_3^{(k)}(q, a)$  and  $C_4^{(k)}(q, a) \in (0, \infty)$  such that*

$$\tilde{H}_a^{(k)}(t, u; f) \leq \left( C_3^{(k)}(q, a) + C_4^{(k)}(q, a) (t-u)^{\frac{-kq-q+2}{2q}} \right) E[|f_u|^p | \mathcal{G}_u^W]^{\frac{1}{p}}$$

for any  $f \in \mathcal{L}^p$ ,  $t > u > 0$ . Note that  $\tilde{H}_a^{(k)}$  is defined in Equation (13).

(2) Let  $k = 0, 1, 2$  and  $p > \frac{4}{3-k}$ . Then there are some  $C_{5,1}^{(k)}(q, a)$  and  $C_{6,1}^{(k)}(q, a) \in (0, \infty)$  such that

$$\int_0^t \tilde{H}_a^{(k)}(t, u; f) du \leq \left( C_{5,1}^{(k)}(q, a) t^{\frac{1}{q}} + C_{6,1}^{(k)}(q, a) t^{\frac{-kq-q+4}{2q}} \right) \left( \int_0^t E[|f_u|^p] du \right)^{\frac{1}{p}},$$

for any  $t > 0$ ,  $f \in \mathcal{L}^p$ .

(3) Let  $k = 0, 1$ ,  $p > \frac{3}{2-k}$ . There are some  $C_{5,2}^{(k)}(q, a)$  and  $C_{6,2}^{(k)}(q, a) \in (0, \infty)$  such that

$$\int_0^t \tilde{H}_a^{(k)}(t, u; f)^2 du \leq \left( C_{5,2}^{(k)}(q, a) t^{\frac{1}{q}} + C_{6,2}^{(k)}(q, a) t^{\frac{-kq-q+3}{q}} \right) \left( \int_0^t E[|f_u|^{2p}] du \right)^{\frac{1}{p}},$$

for any  $t > 0$ ,  $f \in \mathcal{L}^{2p}$ .

(4) Let  $s \in [0, T]$ . There is some  $\hat{C}_1(T, q, a) \in (0, \infty)$  such that

$$E\left[\int_0^s |\bar{H}_a(t; f)| dt\right] \leq \hat{C}_1(T, q, a) \left( \int_0^s E[|f_u|^p] du \right)^{\frac{1}{p}},$$

for any  $f \in \mathcal{L}^p$ ,  $p > 4$ . Note that  $\bar{H}_a^{(k)}$  is defined in Equation (5).

(5) Let  $0 \leq s_0 < s_1$  and  $\xi$  be a bounded  $\mathcal{F}_{s_0}$ -measurable random variable. Then, we have

$$\int_0^s \hat{H}_a^{(2)}(r; \xi 1_{(s_0, s_1]}(\cdot)) dr = 2\hat{H}_a^{(0)}(s; \xi 1_{(s_0, s_1]}(\cdot)).$$

Note that  $\hat{H}_a^{(k)}$  is defined in Equation (4).

*Proof.* (1) By Proposition 2.1, Hölder's inequality and a property of convex function, we have

$$\begin{aligned} & |\tilde{H}_a^{(k)}(t, u; f)| \\ & \leq E[1_{\{\tau^a > u\}}] \left| \frac{\partial^k g}{\partial x^k}(t - u, B_u^a) \right|^q |\mathcal{G}_u^W|^{\frac{1}{q}} E[|f_u|^p |\mathcal{G}_u^W|^{\frac{1}{p}}] \\ & \leq \left( C_1^{(k)}(a) + C_2^{(k)}(q, a) (t - u)^{\frac{-kq-q+2}{2}} \right)^{\frac{1}{q}} \cdot E[|f_u|^p |\mathcal{G}_u^W|^{\frac{1}{p}}] \\ & \leq \left( C_3^{(k)}(q, a) + C_4^{(k)}(q, a) (t - u)^{\frac{-kq-q+2}{2q}} \right) E[|f_u|^p |\mathcal{G}_u^W|^{\frac{1}{p}}], \end{aligned}$$

where

$$C_3^{(k)}(q, a) = 2^{\frac{1}{q}} C_1^{(k)}(a)^{\frac{1}{q}}, \quad C_4^{(k)}(q, a) = 2^{\frac{1}{q}} C_2^{(k)}(q, a)^{\frac{1}{q}}.$$



Next, we will show assertion (2) and (3). Let  $m = 1, 2$ .

$$\begin{aligned}
& \int_0^t \tilde{H}_a^{(k)}(t, u; f)^m du \\
& \leq \int_0^t \left( C_3^{(k)}(q, a) + C_4^{(k)}(q, a)(t - u)^{\frac{-kq - q + 2}{2q}} \right)^{mq} E[|f_u|^p | \mathcal{G}_u^W]^{\frac{m}{p}} du \\
& \leq \left\{ \int_0^t \left( C_3^{(k)}(q, a) + C_4^{(k)}(q, a)(t - u)^{\frac{-kq - q + 2}{2q}} \right)^{mq} du \right\}^{\frac{1}{q}} \left( \int_0^t E[|f_u|^p | \mathcal{G}_u^W]^m du \right)^{\frac{1}{p}} \\
& \leq 2^m \left( \int_0^t (C_3^{(k)}(q, a)^{mq} + C_4^{(k)}(q, a)^{mq}(t - u)^{\frac{(-kq - q + 2)m}{2}}) du \right)^{\frac{1}{q}} \left( \int_0^t E[|f_u|^p | \mathcal{G}_u^W]^m du \right)^{\frac{1}{p}}.
\end{aligned}$$

If  $m = 1$  and  $p > \frac{4}{3-k}$ , or if  $m = 2$  and  $p > \frac{3}{2-k}$ , we have  $p > \frac{2+2m}{2+m-mk}$  and  $\frac{(-kq - q + 2)m}{2} > -1$ . Then we have

$$\begin{aligned}
& \int_0^t \left( C_3^{(k)}(q, a)^{mq} + C_4^{(k)}(q, a)^{mq}(t - u)^{\frac{(-kq - q + 2)m}{2}} \right) du \\
& \leq C_3^{(k)}(q, a)^{mq} t + C_4^{(k)}(q, a)^{mq} \frac{2}{|(-kq - q + 2)m + 2|} t^{\frac{(-kq - q + 2)m + 2}{2}}.
\end{aligned}$$

Then we have the following for  $f \in \mathcal{L}^{mp}$ .

$$\begin{aligned}
& \int_0^t \tilde{H}_a^{(k)}(t, u; f)^m du \\
& \leq (C_{5,m}^{(k)}(q, a)t^{\frac{1}{q}} + C_{6,m}^{(k)}(q, a)t^{\frac{(-kq - q + 2)m + 2}{2q}}) \left( \int_0^t E[|f_u|^{mp}] du \right)^{\frac{1}{p}},
\end{aligned}$$

where

$$C_{5,m}^{(k)}(q, a) = 2^{\frac{mq+m}{q}} C_1^{(k)}(a)^{\frac{m}{q}}, \quad C_{6,m}^{(k)}(q, a) = \frac{2^{\frac{mq+m}{q}}}{|(-kq - q + 2)m + 2|^{\frac{1}{q}}} C_2^{(k)}(q, a)^{\frac{m}{q}}.$$

(4) We can see that  $H_a^{(k)}(t, f)$ ,  $k = 0, 1, 2$ , are well defined for  $f \in \mathcal{L}^{\frac{4}{3-k}+}$  by Assertion (2). Then  $\bar{H}_a(t; f)$  is well defined for  $p \in \mathcal{L}^{4+}$ . Since  $p > 4$  and  $\frac{-3q+4}{2q} > 0$ , Assertion (1) implies

$$\begin{aligned}
& E \left[ \int_0^s |\bar{H}_a(t; f)| dt \right] \\
& \leq E \left[ e^{\int_0^s \lambda_a(r) dr} \int_0^T \left( \hat{H}_a^{(2)}(t; f) + 2\lambda_a(t) \hat{H}_a^{(0)}(t; f) \right) dt \right] \times \left( \int_0^s E[|f_u|^p] du \right)^{\frac{1}{p}} \\
& \leq \hat{C}_1(T, q, a) \left( \int_0^s E[|f_u|^p] du \right)^{\frac{1}{p}},
\end{aligned}$$

where

$$\begin{aligned} & \hat{C}_1(s, q, a) \\ &= e^{\int_0^s \lambda_a(r) dr} \left\{ \int_0^s \left( C_3^{(2)}(a) \frac{1}{q} t^{\frac{1}{q}} + C_4^{(2)}(q, a) \frac{1}{q} t^{\frac{-3q+4}{2q}} \right) + 2\lambda_a(t) \left( C_3^{(0)}(a) \frac{1}{q} t^{\frac{1}{q}} + C_4^{(0)}(q, a) \frac{1}{q} t^{\frac{-q+4}{2q}} \right) dt \right\}. \end{aligned}$$

(5) Since  $1_{\{\tau^a > u\}} \int_u^s \frac{\partial g}{\partial r}(r - u, B_u^a) dr = 1_{\{\tau^a > u\}} g(s - u, B_u^a)$ , we have the following.

$$\begin{aligned} & \int_0^s \hat{H}_a^{(2)}(r; \xi 1_{(s_0, s_1]}(\cdot)) dr \\ &= \int_0^s \left( \int_0^r H_a^{(2)}(r, u; \xi 1_{(s_0, s_1]}(\cdot)) du \right) dr \\ &= \int_0^s \left( \int_u^s H_a^{(2)}(r, u; \xi 1_{(s_0, s_1]}(\cdot)) dr \right) du \\ &= \int_0^s \left( \int_u^s E[1_{\{\tau^a > u\}} \xi 1_{(s_0, s_1]}(u) \frac{\partial^2 g}{\partial x^2}(r - u, B_u^a) | \mathcal{G}_u^W] dr \right) du \\ &= 2 \int_0^s \left( \int_u^s E[1_{\{\tau^a > u\}} \xi 1_{(s_0, s_1]}(u) \frac{\partial g}{\partial r}(r - u, B_u^a) | \mathcal{G}_u^W] dr \right) du \\ &= 2 \hat{H}_a^{(0)}(s; \xi 1_{(s_0, s_1]}(\cdot)). \end{aligned}$$

Note that the last equation holds by Assertion (2) ■

**Proposition 2.3** *Let  $T > 0$ ,  $p > 3$ ,  $q = \frac{p}{p-1}$ . Then  $\bar{U}_a$  is well defined, for any  $f \in \mathcal{L}^{6+}$  and there are  $\tilde{C}_1(q, a, T)$ ,  $\tilde{C}_2(a, T) \in (0, \infty)$  such that*

$$E\left[\int_0^T \left(\int_0^t \bar{U}_a(t, u; f)^2 du\right) dt\right] \leq \tilde{C}_1(q, a, T) \left(\int_0^T E[|f_u|^{2p}] du\right) dt^{\frac{1}{p}} + \tilde{C}_2(a, T) E\left[\int_0^T f_u^2 du\right]$$

for any  $f \in \mathcal{L}^{6+}$ . Note that  $\bar{U}$  is given by Equation(7).

*Proof.* Because  $0 \leq \Phi(t - s, B_s^a) \leq 1$ , for any  $f \in \mathcal{L}^{6+}$ , we have

$$\begin{aligned} & \int_0^t E[U_a(t, u; f)^2] du \\ &\leq \int_0^t E[E[1_{\{\tau^a > s\}} f_u (2\Phi(t - u, B_u^a) - 1) | \mathcal{G}_u^W]^2] du \\ &\leq \int_0^t E[f_u^2 (2\Phi(t - u, B_u^a) - 1)^2] du \leq \int_0^t f_u^2 du. \end{aligned}$$

By the above evaluation and Proposition 2.2 (2), we have

$$\begin{aligned}
& E\left[\int_0^T \left(\int_0^t \bar{U}_a(t, u; f)^2 du\right) dt\right] \\
&= E\left[\int_0^T \left(\int_0^t e^{2\int_0^t \lambda_a(r)dr} (H_a^{(1)}(t, u; f) + \lambda_a(t)^2 U_a(t, u; f))^2 du\right) dt\right] \\
&\leq 2e^{2\int_0^T \lambda_a(r)dr} E\left[\int_0^T \int_0^t \left(\tilde{H}_a^{(1)}(t, u; f)^2 du\right) dt + \int_0^T \lambda_a(t)^2 \left(\int_0^t |U_a(t, u; f)| du\right)^2 dt\right] \\
&\leq 2e^{2\int_0^T \lambda_a(r)dr} \int_0^T \left((C_{5,2}^{(1)}(q, a)t + C_{6,2}^{(1)}(q, a)t^{-2q+3})^{\frac{1}{q}} \lambda_a(t)^2 \left(\int_0^t E[|f_u|^p]^2 du\right)^{\frac{1}{p}}\right) dt \\
&+ 2Te^{2\int_0^T \lambda_a(r)dr} \left(\sup_{0 \leq t \leq T} \lambda_a(t)^2\right) E\left[\int_0^T f_u^2 du\right].
\end{aligned}$$

For a part of first term, we have

$$\begin{aligned}
& \int_0^T ((C_{5,2}^{(1)}(q, a)t + C_{6,2}^{(1)}(q, a)t^{-2q+3})^{\frac{1}{q}} \left(\int_0^t E[|f_u|^{2p}] du\right)^{\frac{1}{p}}) dt \\
&\leq \left(\int_0^T (C_{5,2}^{(1)}(q, a)t + C_{6,2}^{(1)}(q, a)t^{-2q+3}) dt\right)^{\frac{1}{q}} \left(\int_0^T \left(\int_0^t E[|f_u|^{2p}] du\right) dt\right)^{\frac{1}{p}}.
\end{aligned}$$

Note that  $U$  is defined in Equation (6). Then we have the assertion where

$$\begin{aligned}
& \tilde{C}_1(q, a, T) \\
&= 2e^{2\int_0^T \lambda_a(r)dr} \left(\int_0^T (C_{5,2}^{(1)}(q, a)t + C_{6,2}^{(1)}(q, a)t^{-2q+3}) dt\right)^{\frac{1}{q}} \left(\int_0^T (\lambda_a(t)^2 \int_0^t E[|f_u|^{2p}] du) dt\right)^{\frac{1}{p}}
\end{aligned}$$

and

$$\tilde{C}_2(a, T) = 2Te^{2\int_0^T \lambda_a(r)dr} \left(\sup_{0 \leq t \leq T} \lambda_a(t)^2\right).$$

■

### 3 Representation theorem

We saw that some integrals are well defined under the conditions in Section 2. In this section, we prove Theorem 1.1, which is the representation theorem under  $\mathcal{F}_t^W$ . For  $x, y \geq 0$  and  $t > 0$ , let

$$g_0(t, x, y) = g(t, y - x) - g(t, y + x) = g(t, y - x)(1 - e^{-2xy/t}) \quad (15)$$

where  $g(t, x)$  and  $\Phi(t, x)$  are the density and distribution, respectively, of the Brownian motion  $B_t$ . These are given by Equation (2).

First, we will present a representation theorem for  $E[\int_0^t \cdot dB_s | \mathcal{F}_t^W]$  which corresponds to Theorem 1.1(1).

**Lemma 3.1** *Let  $t > u > 0$  and  $\xi$  be a bounded  $\mathcal{B}_u$ -measurable random variable. Then we have*

$$E[\xi | \mathcal{G}_\infty^W] = E[\xi | \mathcal{G}_u^W] \text{ and } E[\xi(B_t - B_u) | \mathcal{G}_\infty^W] = 0.$$

*Proof.* Let  $h_0$  be a bounded  $\mathcal{G}_u^W$ -measurable random variable and  $h_1$  be a bounded  $\sigma\{W(s) - W(u); s \geq u\}$  measurable random variable. Then

$$\begin{aligned} & E[\xi h_0 h_1] \\ &= E[\xi h_0 E[h_1 | \mathcal{B}_u]] = E[\xi h_0] E[h_1] \\ &= E[E[\xi | \mathcal{G}_u^W] h_0] E[h_1] = E[E[\xi | \mathcal{G}_u^W] h_0 h_1] \end{aligned}$$

and

$$E[\xi(B_t - B_u) h_0 h_1] = E[\xi h_0] E[(B_t - B_u) h_1] = 0.$$

So we have our assertion. ■

**Proposition 3.2** *Let  $0 \leq s_0 < s_1$ ,  $\xi$  be a bounded  $\mathcal{B}_{s_0}$ -measurable random variable. Then, we have the following for  $t \geq 0$ ,*

$$\begin{aligned} & E[\xi 1_{\{\tau^a > t\}} (B_{s_1}^a - B_{s_0}^a)] \\ &= - \int_t^\infty \left( \int_{s_0}^{s_1} 1_{\{u < r\}} E[\xi 1_{\{\tau^a > u\}}] \frac{\partial^2 g}{\partial x^2}(r - u, B_u^a) du \right) dr. \end{aligned} \quad (16)$$

*Proof.* Let

$$\varphi(s, x, t) = \int_0^\infty \int_0^\infty (y - x) g_0(s, x, y) g_0(t, y, z) dy dz, \quad x > 0, s, t > 0.$$

Note that  $g_0$  is defined in Equation (15). At first, let us think about the case  $t > s_1$ . Then we have

$$1_{\{\tau^a > s_0\}} E[1_{\{\tau^a > t\}} (B_{s_1}^a - B_{s_0}^a) | \mathcal{B}_{s_0}] = 1_{\{\tau^a > s_0\}} \varphi(s_1 - s_0, B_{s_0}^a, t - s_1).$$

Then

$$E[\xi 1_{\{\tau^a > t\}} (B_{s_1}^a - B_{s_0}^a)] = E[\xi 1_{\{\tau^a > s_0\}} \varphi(s_1 - s_0, B_{s_0}^a, t - s_1)].$$

Note that

$$|\varphi(s, x, t)| \leq \int_{-\infty}^\infty \int_{-\infty}^\infty |y - x| g(s, x - y) g(t, y - z) dz dy$$

$$= \int_{-\infty}^{\infty} |y - x| g(s, x - y) dy = E[|B_s|] = \sqrt{\frac{2s}{\pi}}. \quad (17)$$

Since

$$d_s \varphi(s_1 - s, B_s^a, r) = \left(-\frac{\partial}{\partial s} + \frac{1}{2} \frac{\partial^2}{\partial x^2}\right) \varphi(s_1 - s, B_s^a, r) ds + \frac{\partial \varphi}{\partial x}(s_1 - s, B_s^a, r) dB_s^a$$

and

$$\begin{aligned} & \left(-\frac{\partial}{\partial s} + \frac{1}{2} \frac{\partial^2}{\partial x^2}\right) \varphi(s, x, r) \\ &= - \int_0^\infty \int_0^\infty \frac{\partial g_0}{\partial x}(s, x, y) g_0(r, y, z) dy dz = - \int_0^\infty \frac{\partial g_0}{\partial x}(s + r, x, z) dz \\ &= - \int_0^\infty \left( \frac{\partial g}{\partial x}(s + r, x - z) - \frac{\partial g}{\partial x}(s + r, x + z) \right) dz = -2g(s + r, x), \quad x > 0, \quad s, r > 0, \end{aligned}$$

we have

$$\begin{aligned} & 1_{\{\tau^a > s_0\}} (\varphi(s_1 - s \wedge \tau^a, B_{s \wedge \tau^a}^a, t - s_1) - \varphi(s_1 - s_0, B_{s_0}^a, t - s_1)) \\ &= -2 \int_{s_0}^{s \wedge \tau^a} g(t - u, B_u^a) du + \int_{s_0}^{s \wedge \tau^a} \frac{\partial \varphi}{\partial x}(s_1 - u, B_u^a, t - s_1) dB_u^a, \quad s \in [s_0, s_1]. \end{aligned}$$

As  $2 \frac{\partial g}{\partial t} = \frac{\partial^2 g}{\partial x^2}$ , we have

$$\begin{aligned} & E[\xi 1_{\{\tau^a > t\}} (B_{s_1}^a - B_{s_0}^a)] \\ &= E[\xi 1_{\{\tau^a > s_0\}} \varphi(s_1 - s \wedge \tau^a, B_{s \wedge \tau^a}^a, t - s_1)] \\ &+ 2E[\xi 1_{\{\tau^a > s_0\}} \left( \int_{s_0}^s 1_{\{\tau^a > u\}} g(t - u, B_u^a) du \right)], \quad s \in [s_0, s_1]. \end{aligned}$$

Since  $\varphi(s, 0, t) = 0$  and  $\varphi(s, x, t) \rightarrow 0$ ,  $s \downarrow 0$ , we have

$$\lim_{s \rightarrow s_1} E[\xi 1_{\{\tau^a > s_0\}} \varphi(s_1 - s \wedge \tau^a, B_{s \wedge \tau^a}^a, t - s_1)] \rightarrow 0$$

by Equation (17) and the bounded convergence theorem. Then we have

$$\begin{aligned} & E[\xi 1_{\{\tau^a > t\}} (B_{s_1}^a - B_{s_0}^a)] \\ &= -2 \int_{s_0}^{s_1} E[\xi 1_{\{\tau^a > u\}} g(t - u, B_u^a)] dr du \\ &= -2 \int_{s_0}^{s_1} E[\xi \int_t^\infty \frac{\partial g}{\partial r}(r - u, B_u^a) dr] du \\ &= - \int_t^\infty \left( \int_{s_0}^{s_1} 1_{\{u < r\}} E[\xi 1_{\{\tau^a > u\}} \frac{\partial^2 g}{\partial x^2}(r - u, B_u^a)] du \right) dr \end{aligned}$$

for any  $t > s_1$ . By taking  $t \downarrow s_1$ , we also have our assertion for  $t = s_1$ .  
Second, let us think of the case  $t \in (s_0, s_1]$ .

$$\begin{aligned} E[\xi 1_{\{\tau^a > t\}}(B_{s_1}^a - B_{s_0}^a)] &= E[\xi 1_{\{\tau^a > t\}} E[(B_{s_1}^a - B_{s_0}^a) | \mathcal{B}_t]] \\ &= E[\xi 1_{\{\tau^a > t\}}(B_t^a - B_{s_0}^a)] = - \int_t^\infty \left( \int_{s_0}^t 1_{\{u < r\}} E[\xi 1_{\{\tau^a > u\}} \frac{\partial^2 g}{\partial x^2}(r - u, B_u^a)] du \right) dr. \end{aligned}$$

Let  $p > 4$  and  $q = \frac{p}{p-1}, r > u \geq 0$ . Then we have

$$\begin{aligned} & E[|1_{\{\tau^a > u\}} \xi \frac{\partial^2 g}{\partial x^2}(r - u, B_u^a)|] \\ & \leq E[|1_{\{\tau^a > u\}} \xi|^p]^{\frac{1}{p}} E[|1_{\{\tau^a > u\}} \frac{\partial^2 g}{\partial x^2}(r - u, B_u^a)|^q]^{\frac{1}{q}} \\ & \leq E[|1_{\{\tau^a > u\}} \xi|^p]^{\frac{1}{p}} E[(C_1^{(2)}(a) + C_2^{(2)}(2, a)(r - u)^{\frac{-3q+2}{2}}]^{\frac{1}{q}} \end{aligned} \quad (18)$$

by Proposition 2.1. We have the following by Lemma 3.1.

$$\begin{aligned} & \int_t^\infty \left( \int_t^{s_1} 1_{\{u < r\}} E[1_{\{\tau^a > u\}} \xi \frac{\partial^2 g}{\partial x^2}(r - u, B_u^a)] du \right) dr \\ &= 2 \int_t^{s_1} \left( \int_t^\infty 1_{\{u < r\}} E[1_{\{\tau^a > u\}} \xi \frac{\partial g}{\partial r}(r - u, B_u^a)] dr \right) du \\ &= -2 \int_t^{s_1} (E[1_{\{\tau^a > u\}} \xi \int_u^\infty \frac{\partial g}{\partial r}(r - u, B_u^a)] dr) du = 0. \end{aligned}$$

Note that since  $\frac{-3q+2}{2} > -1$  and by Equation(18), we can use Fubini's Theorem in the above equation. So we have Equation (16) for  $t \in (s_0, s_1]$ .

When  $t \in [0, s_0]$ ,

$$E[\xi 1_{\{\tau^a > t\}}(B_{s_1}^a - B_{s_0}^a)] = E[\xi 1_{\{\tau^a > t\}} E[B_{s_1}^a - B_{s_0}^a | \mathcal{B}_{s_0}]] = 0.$$

So we see Equation (16) is valid for  $t \geq 0$ . ■

**Proposition 3.3** *Let  $0 \leq s_0 < s_1$ ,  $t > 0$ , and  $\xi$  be a bounded  $\mathcal{F}_{s_0}$ -measurable random variable. Then, we have*

$$E[\xi 1_{\{\tau^a > s_0\}} 1_{\{\tau^a > t\}}] = - \int_{s_0 \vee t}^\infty E[1_{\{\tau^a > s_0\}} \xi \frac{\partial g}{\partial x}(r - s_0, B_{s_0}^a)] dr.$$

*Proof.* We assume that  $t > s_0$ , then we have

$$E[\xi 1_{\{\tau^a > t\}}] = E[\xi 1_{\{\tau^a > s_0\}} E[1_{\{\tau^a > t\}} | \mathcal{B}_{s_0}]] = E[\xi 1_{\{\tau^a > s_0\}} \left( \int_0^\infty g_0(t - s_0, B_{s_0}^a, y) dy \right)].$$

For  $x > 0$  and  $t > 0$ , we have

$$\begin{aligned} \int_0^\infty g_0(t, x, y) dy &= - \int_0^\infty \left( \int_t^\infty \frac{\partial g_0}{\partial s}(s, x, y) ds \right) dy \\ &= - \frac{1}{2} \int_t^\infty \left( \int_0^\infty \frac{\partial^2 g_0}{\partial y^2}(s, x, y) dy \right) ds = \frac{1}{2} \int_t^\infty \frac{\partial g_0}{\partial y}(s, x, 0) ds = - \int_t^\infty \frac{\partial g}{\partial x}(s, x) ds. \end{aligned}$$

Considering Equation (14) in Proposition 2.1, we have

$$E[\xi 1_{\{\tau^a > s_0\}} 1_{\{\tau^a > t\}}] = - \int_{s_0 \vee t}^\infty E[\xi 1_{\{\tau^a > s_0\}} \frac{\partial g}{\partial x}(r - s_0, B_{s_0}^a)] dr.$$

■

**Proposition 3.4** *Let  $0 \leq s_0 < s_1$ ,  $\xi$  be a bounded  $\mathcal{F}_{s_0}$ -measurable random variable, and  $v : [0, \infty) \rightarrow \mathbf{R}$  be a bounded Borel measurable function. Then we have the following.*

(1)

$$E[\xi(B_{s_1}^a - B_{s_0}^a)v(\tau^a)] = - \int_0^\infty v(r) \left( \int_{s_0}^{s_1} 1_{\{u < r\}} E[\xi 1_{\{\tau^a > u\}} \frac{\partial^2 g}{\partial x^2}(r - u, B_u^a)] du \right) dr.$$

$$(2) \quad E[\xi 1_{\{\tau^a > s_0\}} v(\tau^a)] = - \int_{s_0}^\infty v(r) E[1_{\{\tau^a > s_0\}} \xi \frac{\partial g}{\partial x}(r - s_0, B_{s_0}^a)] dr.$$

$$(3) \quad E[\xi(B_{s_1}^a - B_{s_0}^a) | \mathcal{F}_\infty^W] = - \int_0^\infty \gamma_a^{-1}(r) \hat{H}_a^{(2)}(r; \xi 1_{(s_0, s_1]}(\cdot)) dN_r^a.$$

$$(4) \quad E[\xi(B_{s_1}^a - B_{s_0}^a) | \mathcal{F}_t^W] = - \int_0^t \bar{H}_a(s; \xi 1_{(s_0, s_1]}(\cdot)) \lambda_a(s)^{-1} dM_s^a, \quad t > 0.$$

*Proof.* (1) For  $v = 1_{[t, \infty)}$ , Assertion (1) is valid by Proposition 3.2. Let  $\mathcal{V}$  be the collection of bounded measurable functions  $v$  which satisfy Assertion (1). Then  $\mathcal{V}$  is a vector space. In addition, if  $\{v_n\}_{n \in \mathbf{N}}$  is an increasing sequence of non-negative functions in  $\mathcal{V}$  and if  $\lim_{n \rightarrow \infty} v_n$  exists and bounded then  $\lim_{n \rightarrow \infty} v_n \in \mathcal{V}$ . Let  $\mathcal{A} = \{A \subset \mathbf{R}; 1_A \in \mathcal{V}\}$  then  $(t, \infty) \in A$  for each  $t > 0$ .  $\mathcal{A}$  is  $\pi$ -system by the monotone convergence Theorem and  $\mathcal{A}' = \{(t, \infty); t > 0\} \in \mathcal{A}$  is  $\pi$ -system. Then we have our assertion by the monotone class theorem.

(2) By the same way with Assertion (1), we see that this assertion is valid for any bounded Borel measurable function  $v : (0, \infty) \rightarrow \mathbf{R}$  using Proposition 3.3. This completes the proof of Assertion.

(3) Let  $h_0$  be a bounded  $\mathcal{G}_{s_0}^W$ -measurable Borel function and  $h_1$  be a bounded  $\sigma\{W_t - W_{s_0}; t > s_0\}$ -measurable function. Note that  $\mathcal{B}_{s_0} \vee \mathcal{G}_\infty^B$  and  $\sigma\{W_t - W_{s_0}; t > s_0\}$  are independent. By Lemma 3.1 and Proposition 3.4 (1), we

have

$$\begin{aligned}
& E[\xi(B_{s_1}^a - B_{s_0}^a)h_0h_1] = -E[h_0\xi(B_{s_1}^a - B_{s_0}^a)]E[h_1] \\
& = -\left(\int_0^\infty \left(\int_{s_0}^{s_1} 1_{\{u < r\}} E[h_0\xi 1_{\{\tau^a > u\}}] \frac{\partial^2 g}{\partial x^2}(r-u, B_u^a) du\right) dr\right) E[h_1] \\
& = -\left(\int_0^\infty \left(\int_{s_0}^{s_1} 1_{\{u < r\}} E[h_0h_1 E[\xi 1_{\{\tau^a > u\}}] \frac{\partial^2 g}{\partial x^2}(r-u, B_u^a) | \mathcal{G}_\infty^W] du\right) dr\right) \\
& = -E[h_0h_1 \left(\int_0^\infty \left(\int_0^\infty 1_{\{u < r\}} E[\xi 1_{(s_0, s_1]}(u) 1_{\{\tau^a > u\}}] \frac{\partial^2 g}{\partial x^2}(r-u, B_u^a) | \mathcal{G}_\infty^W] du\right) dr\right) \\
& = -E[h_0h_1 \gamma_a(\tau^a)^{-1} \int_0^\infty 1_{\{\tau^a > u\}} E[\xi 1_{(s_0, s_1]}(u) 1_{\{\tau^a > u\}}] \frac{\partial^2 g}{\partial x^2}(\tau^a - u, B_u^a) | \mathcal{G}_u^W] du] \\
& = -E[h_0h_1 \gamma_a(\tau^a)^{-1} \hat{H}_a^{(2)}(\tau^a; \xi 1_{(s_0, s_1]}(\cdot))] = -E[h_0h_1 \int_0^\infty \gamma_a(r)^{-1} \hat{H}_a^{(2)}(r; \xi 1_{(s_0, s_1]}(\cdot)) dN_r^a].
\end{aligned}$$

Then we have the assertion.

(4) We note that  $\hat{H}_a^{(2)}(t; \xi 1_{(s_0, s_1]}(\cdot)) = 0$  for  $t \leq s_0$  and

$$E[\xi(B_{s_1}^a - B_{s_0}^a) | \mathcal{F}_{s_0}^W] = E[E[\xi(B_{s_1}^a - B_{s_0}^a) | \mathcal{B}_{s_0}]] | \mathcal{F}_{s_0}^W] = 0.$$

Then we have

$$E[\xi(B_{s_1}^a - B_{s_0}^a) | \mathcal{F}_t^W] = 0 = \int_0^t \bar{H}_a(s; \xi 1_{(s_0, s_1]}(\cdot)) \lambda_a(s)^{-1} dM_s^a, \quad t \leq s_0.$$

By Lemma 3.1, we have

$$\int_0^\infty \hat{H}_a^{(2)}(s; \xi 1_{(s_0, s_1]}(\cdot)) ds = E[\xi(B_{s_1}^a - B_{s_0}^a) | \mathcal{G}_\infty^W] = 0$$

and then

$$E\left[\int_t^\infty \hat{H}_a^{(2)}(s; \xi 1_{(s_0, s_1]}(\cdot)) ds | \mathcal{G}_t^W\right] = -\int_0^t \hat{H}_a^{(2)}(s; \xi 1_{(s_0, s_1]}(\cdot)) ds. \quad (19)$$



By Assertion (3) and Equation (19), we see that

$$\begin{aligned}
& E[\xi(B_{s_1}^a - B_{s_0}^a) | \mathcal{F}_t^W] \\
&= -E\left[\int_0^\infty \gamma_a^{-1}(r) \hat{H}_a^{(2)}(r; \xi 1_{(s_0, s_1]}(\cdot)) dN_r^a | \mathcal{F}_t^W\right] \\
&= -\int_0^t \gamma_a^{-1}(r) \hat{H}_a^{(2)}(r; \xi 1_{(s_0, s_1]}(\cdot)) dN_r^a - E\left[\int_t^\infty \gamma_a^{-1}(r) \hat{H}_a^{(2)}(r; \xi 1_{(s_0, s_1]}(\cdot)) dN_r^a | \mathcal{F}_t^W\right] \\
&= -\int_0^t e^{\int_0^s \lambda_a(r) dr} \hat{H}_a^{(2)}(s; \xi 1_{(s_0, s_1]}(\cdot)) \lambda_a(s)^{-1} dM_s^a \\
&= -\int_0^t e^{\int_0^s \lambda_a(r) dr} \hat{H}_a^{(2)}(s; \xi 1_{(s_0, s_1]}(\cdot)) (1 - N_s^a) ds \\
&+ e^{\int_0^t \lambda_a(r) dr} (1 - N_t^a) \int_0^t \hat{H}_a^{(2)}(s; \xi 1_{(s_0, s_1]}(\cdot)) ds. \tag{20}
\end{aligned}$$

Note that  $\gamma_a$  is defined in Equation (1). We also note that  $e^{\int_0^t \lambda_a(r) dr} (1 - N_t^a) = 1 - \int_0^t e^{\int_0^s \lambda_a(r) dr} dM_s^a$ . We now see that

$$\begin{aligned}
& e^{\int_0^t \lambda_a(r) dr} (1 - N_t^a) \int_0^t \hat{H}_a^{(2)}(s; \xi 1_{(s_0, s_1]}(\cdot)) ds \\
&= \int_0^t \hat{H}_a^{(2)}(s; \xi 1_{(s_0, s_1]}(\cdot)) ds - \left(\int_0^t e^{\int_0^s \lambda_a(r) dr} dM_s^a\right) \left(\int_0^t \hat{H}_a^{(2)}(s; \xi 1_{(s_0, s_1]}(\cdot)) ds\right) \\
&= -\int_0^t e^{\int_0^s \lambda_a(r) dr} \left(\int_0^s \hat{H}_a^{(2)}(r; \xi 1_{(s_0, s_1]}(\cdot)) dr\right) dM_s^a \\
&+ \int_0^t \hat{H}_a^{(2)}(s; \xi 1_{(s_0, s_1]}(\cdot)) ds + \int_0^t \left(\hat{H}_a^{(2)}(s; \xi 1_{(s_0, s_1]}(\cdot)) \int_0^s e^{\int_0^r \lambda_a(u) du} dM_r^a\right) ds.
\end{aligned}$$

Then, we have the following for  $t \geq s_0$ ,

$$\begin{aligned}
& E[\xi(B_{s_1}^a - B_{s_0}^a) | \mathcal{F}_t^W] = -\int_0^t e^{\int_0^s \lambda_a(r) dr} \hat{H}_a^{(2)}(s; \xi 1_{(s_0, s_1]}(\cdot)) \lambda_a(s)^{-1} dM_s^a \\
&+ \int_0^t \left(\int_0^s \hat{H}_a^{(2)}(r; \xi 1_{(s_0, s_1]}(\cdot)) dr\right) d(e^{\int_0^s \lambda_a(r) dr} (1 - N_{s-}^a)) \\
&= -\int_0^t e^{\int_0^s \lambda_a(r) dr} \left(\hat{H}_a^{(2)}(s; \xi 1_{(s_0, s_1]}(\cdot)) + \lambda_a(s) \int_0^s \hat{H}_a^{(2)}(r; \xi 1_{(s_0, s_1]}(\cdot)) dr\right) \lambda_a(s)^{-1} dM_s^a.
\end{aligned}$$

Finally, we have Assertion by Proposition 2.2 (4).  $\blacksquare$

Let  $\tilde{\mathcal{L}}_0$  be the space of progressively measurable processes  $\varphi_t$  for which there exist  $\mathcal{B}_{s_k}$ -measurable bounded random variables  $\xi_{s_k}$  such that

$$\varphi_t = \sum_{k=0}^{n-1} \xi_{s_k} 1_{(s_k, s_{k+1}]}(t), \quad t \geq 0,$$

where  $0 \leq s_0 < s_1 < \dots < s_n \leq T$ . For any  $p \geq 1$  and  $f \in \mathcal{L}^p$ , there exist  $f_n \in \tilde{\mathcal{L}}^0$ ,  $n = 1, 2, \dots$ , such that

$$\lim_{n \rightarrow \infty} E\left[\int_0^T |f_n(s, \omega) - f(s, \omega)|^p ds\right] = 0 \quad \text{for any } T > 0.$$

The following gives Theorem 1.1(1).

**Corollary 3.5** *Let  $T > 0$ . Then we have*

$$E\left[\int_0^T f_s dB_s | \mathcal{F}_\infty^W\right] = - \int_0^\infty \gamma_a(s)^{-1} \left( \int_0^\infty H_a^{(2)}(s, u; f 1_{[0, T]}(\cdot)) du \right) dN_s^a$$

for any  $f \in \mathcal{L}^{4+}$  and

$$E\left[\int_0^T f_s dB_s | \mathcal{F}_t^W\right] = - \int_0^t \bar{H}_a(s, f 1_{[0, T]}(\cdot)) \lambda_a(s)^{-1} dM_s^a, \quad t > 0$$

for any  $f \in \mathcal{L}^{4+}$ .

*Proof.* Let  $s_1 > s_0 \geq 0$  and  $\tilde{f}$  be a bounded  $\mathcal{B}_{s_0}$ -measurable function and  $f_t = \tilde{f} 1_{(s_0, s_1]}(t)$ . Then we see that the first and second assertion are valid for  $f \in \tilde{\mathcal{L}}^{(0)}$  by Proposition 3.4 (3) and (4), respectively. We can see that  $\int_0^\infty H_a^{(2)}(s, u; f 1_{[0, T]}(\cdot)) du$  in the first assertion is well defined for any  $f \in \mathcal{L}^{4+}$  by Proposition 2.2 (2). As for the second assertion, let us take  $\{\tilde{\xi}_n\} \in \tilde{\mathcal{L}}_0$  such that

$$\lim_{n \rightarrow \infty} E[|\tilde{\xi}_n(r) - f_r|] = 0 \quad \text{for all } r > 0.$$

Then we have

$$E\left[\int_0^T \tilde{\xi}_n(s) dB_s | \mathcal{F}_t^W\right] = - \int_0^t \bar{H}_a(s, \tilde{\xi}_n 1_{[0, T]}(\cdot)) \lambda_a(s)^{-1} dM_s^a, \quad t > 0,$$

by Proposition 3.4 (4). Since  $\sigma\{W_t; t \geq 0\}$  and  $\sigma\{N_t; t \geq 0\}$  are independent, we have

$$\begin{aligned} & E\left[\int_0^T |(\bar{H}_a(s; \tilde{\xi}_n) - \bar{H}_a(s; f))| \lambda_a(s)^{-1} dN_s^a\right] \\ &= E\left[\int_0^T E[|(\bar{H}_a(s; \tilde{\xi}_n) - \bar{H}_a(s; f))|] \lambda_a(s)^{-1} dN_s^a\right] \\ &= \int_0^T E[|(\bar{H}_a(s; \tilde{\xi}_n - f))|] e^{-\int_0^s \lambda_a(u) du} ds \rightarrow 0, \quad \text{as } n \rightarrow \infty, \text{ for all } T > 0 \end{aligned}$$

by Proposition 2.2 (4). So  $\int_0^t \bar{H}_a(s, f 1_{[0, T]}(\cdot)) \lambda_a(s)^{-1} dM_s$  is well defined, and we have the assertion.  $\blacksquare$

Second, we will state a representation theorem for  $E[\int_0^t \cdot ds | \mathcal{F}_t^W]$ , which corresponds to Theorem 1.1(2).

**Proposition 3.6** *Let  $s > 0$  and  $f$  be a bounded  $\mathcal{F}$ -progressively measurable process. Then, we have the following.*

$$(1) E[f_s | \mathcal{F}_\infty^W] = E[f_s | \mathcal{F}_s^W] 1_{\{\tau^a \leq s\}} - \int_s^\infty \gamma_a(r)^{-1} H_a^{(1)}(r, s; f) dN_r^a. (2) E[f_s | \mathcal{F}_t^W] = E[f_s | \mathcal{F}_s^W] - \int_s^t \gamma_a(r)^{-1} \bar{U}_a(r, s; f) \lambda_a(r)^{-1} dM_r^a, \quad t > s.$$

*Proof.*

(1) Let  $s > 0$ ,  $h_0$  be a bounded  $\mathcal{G}_s^W$ -measurable Borel function,  $h_1$  be a bounded  $\sigma\{W_t - W_s; t > s\}$ -measurable function and  $v : [0, \infty) \rightarrow \mathbf{R}$  be a bounded Borel measurable function. Then we have

$$E[f_s v(\tau^a) h_0 h_1] = E[f_s 1_{\{\tau^a \leq s\}} v(\tau^a) h_0 h_1] + E[h_1] E[f_s 1_{\{\tau^a > s\}} v(\tau^a) h_0]$$

and

$$\begin{aligned} E[f_s 1_{\{\tau^a \leq s\}} v(\tau^a) h_0 h_1] &= E[h_1] E[f_s 1_{\{\tau^a \leq s\}} v(\tau^a) h_0] \\ &= E[h_1] E[E[f_s | \mathcal{F}_s^W] 1_{\{\tau^a \leq s\}} v(\tau^a) h_0] = E[E[f_s | \mathcal{F}_s^W] 1_{\{\tau^a \leq s\}} v(\tau^a) h_0 h_1]. \end{aligned}$$

Since  $\sigma\{W_t; t \geq 0\}$  and  $\sigma\{N_t; t \geq 0\}$  are independent, we have the following by Proposition 3.4 (2),

$$\begin{aligned} &E[h_1] E[f_s 1_{\{\tau^a > s\}} v(\tau^a) h_0] \\ &= E[h_1] \int_s^\infty v(r) E[1_{\{\tau^a > s\}} f_s \frac{\partial g}{\partial x}(r - s, B_s^a) h_0] dr \\ &= -E[h_1] \int_s^\infty v(r) \gamma_a(r) E[\gamma_a(r)^{-1} H_a^{(1)}(r, s; f) h_0] dr \\ &= -\int_s^\infty v(r) \gamma_a(r) E[\gamma_a(r)^{-1} H_a^{(1)}(r, s; f) h_0 h_1] dr \\ &= -E[E[v(r) \gamma_a(r)^{-1} H_a^{(1)}(r, s; f) h_0 h_1] |_{r=\tau^a} 1_{\{\tau^a > s\}}] \\ &= -E[\gamma_a(\tau^a)^{-1} H_a^{(1)}(\tau^a, s; f) 1_{\{\tau^a > s\}} v(\tau^a) h_0 h_1] \\ &= -E[(\int_s^\infty \gamma_a(r)^{-1} H_a^{(1)}(r, s; f) dN_r^a) v(\tau^a) h_0 h_1]. \end{aligned}$$

So we have

$$E[f_s 1_{\{\tau^a > s\}} | \mathcal{F}_\infty^W] = - \int_s^\infty \gamma_a(r)^{-1} H_a^{(1)}(r, s; f) 1_{\{\tau^a > s\}} dN_r^a.$$

Thus we have Assertion.

(2) Note that

$$\begin{aligned} &\frac{\partial}{\partial t} (2\Phi(t - s, x) - 1) \\ &= 2 \int_{-\infty}^x \frac{\partial g}{\partial t}(t - s, y) dy = \int_{-\infty}^x \frac{\partial^2 g}{\partial y^2}(t - s, y) dy = \frac{\partial g}{\partial x}(t - s, x) \end{aligned}$$

and that

$$2\Phi(t-s, x) - 1 = - \int_t^\infty \frac{\partial}{\partial r} (2\Phi(r-s, x) - 1) dr = - \int_t^\infty \frac{\partial g}{\partial x} (r-s, x) dr.$$

Here we note that

$$\lim_{t \rightarrow \infty} \Phi(t-s, x) = \frac{1}{2}, \quad x > 0$$

and

$$\lim_{t \downarrow s} \Phi(t-s, x) = 1, \quad x > 0.$$

Let

$$L_t = 1 - \exp\left(\int_0^t \lambda_a(s) ds\right)(1 - N_t^a).$$

Then we have

$$dL_t = \exp\left(\int_0^t \lambda_a(s) ds\right)(dN_t^a - \lambda_a(t)(1 - N_t^a)dt) = \exp\left(\int_0^t \lambda_a(s) ds\right)dM_t^a.$$

We note that

$$dN_t^a = \exp\left(-\int_0^t \lambda_a(s) ds\right)dL_t + \lambda_a(t)(1 - N_t^a)dt$$

and

$$\begin{aligned} \gamma_a(t)^{-1}dN_t^a &= \lambda_a(t)^{-1}dL_t + \exp\left(\int_0^t \lambda_a(s) ds\right)(1 - N_t^a)dt \\ &= \lambda_a(t)^{-1}dL_t - L_t dt + \exp\left(\int_0^t \lambda_a(s) ds\right)dt. \end{aligned}$$

Then we have

$$\begin{aligned} U_a(t, s, f) &= E[1_{\{\tau^a > s\}} f_s (2\Phi(t-s, B_s^a) - 1) | \mathcal{G}_s^W] \\ &= -E\left[\int_t^\infty 1_{\{\tau^a > s\}} f_s \frac{\partial g}{\partial x}(r-s, B_s^a) | \mathcal{G}_s^W\right] dr \\ &= -\int_t^\infty H_a^{(1)}(r, s; f) dr. \end{aligned}$$

It is obvious that

$$\begin{aligned} &\int_s^\infty H_a^{(1)}(r, s; f) \gamma_a(r)^{-1} dN_r^a \\ &= \int_s^\infty H_a^{(1)}(r, s; f) \lambda_a(r)^{-1} dL_r - \int_s^\infty H_a^{(1)}(r, s; f) L_r dr + \int_s^\infty H_a^{(1)}(r, s; f) e^{\int_0^r \lambda_a(u) du} dr. \end{aligned}$$

Here we note that the third term at the last equation is  $\mathcal{F}_s$ -measurable. And the second term of the above can be described in the following.

$$\begin{aligned}
& - \int_s^\infty H_a^{(1)}(r, s; f) L_r dr \\
&= - \int_s^\infty H_a^{(1)}(r, s; f) \left( \int_s^r dL_u + L_s \right) dr \\
&= - \int_s^\infty \left( \int_0^\infty H_a^{(1)}(r, s; f) dr \right) dL_u - \int_s^\infty H_a^{(1)}(r, s; f) L_s dr \\
&= \int_s^\infty U_a(r, s; f) dL_r + L_s U_a(s+, s; f).
\end{aligned}$$

Here we note that the second term at the last equation is  $\mathcal{F}_s$ -measurable. Then we have

$$\begin{aligned}
& E[f_s | \mathcal{F}_\infty^W] \\
&= \left( E[f_s | \mathcal{F}_s^W] 1_{\{\tau^a \leq s\}} - L_s U_a(s+, s; f) - \int_s^\infty H_a^{(1)}(r, s; f) e^{\int_0^r \lambda_a(s) ds} dr \right) \\
& \quad - \int_s^\infty (H_a^{(1)}(r, s; f) \lambda_a(r)^{-1} + U_a(r, s; f)) dL_r.
\end{aligned}$$

The first three terms are  $\mathcal{F}_s^W$ -measurable and the summation should be equal to  $E[f_s | \mathcal{F}_s^W]$ . The last term is equal to

$$\begin{aligned}
& \int_s^\infty \left\{ \exp \left( \int_0^r \lambda_a(u) du \right) (H_a^{(1)}(r, s; f) + \lambda_a(r) U_a(r, s; f)) \lambda_a(r)^{-1} \right\} dM_r^a \\
&= \int_s^\infty \bar{U}_a(r, s; f) \lambda_a(r)^{-1} dM_r^a.
\end{aligned}$$

Then we have our assertion. ■

The following gives Theorem 1.1(2).

**Proposition 3.7** *Let  $T, t > 0$  and  $f \in \mathcal{L}^{2+}$ . Then, we have*

$$\begin{aligned}
& E \left[ \int_0^t f_s ds | \mathcal{F}_t^W \right] = \int_0^t E[f_s | \mathcal{F}_t^W] ds \\
&= \int_0^t E[f_s 1_{[0, T](s)} | \mathcal{F}_s^W] ds - \int_0^t \left\{ \left( \int_0^r \bar{U}_a(s, r; f 1_{[0, T]}) ds \right) \lambda_a(s)^{-1} \right\} dM_r^a.
\end{aligned}$$

*Proof.* Remember that  $U_a(t, s; f) = E[1_{\{\tau^a > s\}} f_s (2\Phi(t - s, B_s^a) - 1) | \mathcal{G}_s^W]$ . We can see that  $\int_0^t (\int_0^r \bar{U}_a(s, r; f 1_{[0, T]}) ds) \lambda_a(s)^{-1} dM_r^a$  is well defined for any  $f \in \mathcal{L}^{2+}$  by Proposition 2.2 (2). Then we have the assertion by Proposition 2.3 and 3.6. ■

Third, we prove Theorem 1.1 (3) as follows.

**Proposition 3.8** *Let  $s_1 > s_0 \geq 0$ , and  $\xi$  be a bounded  $\mathcal{F}$ -measurable process. Then we have*

$$\begin{aligned} & E\left[\int_0^\infty \xi 1_{(s_0, s_1]}(r) dW_r \middle| \mathcal{F}_\infty^W\right] \\ &= \int_0^\infty E[\xi 1_{(s_0, s_1]}(r) | \mathcal{F}_r^W] dW_r - \int_0^\infty \left\{ \left( \int_0^r \bar{U}_a(r, u; \xi 1_{(s_0, s_1]}(\cdot)) dW_u \right) \lambda_a(r)^{-1} \right\} dM_r^a. \end{aligned}$$

In particular, for any  $T > 0$  and  $f \in \mathcal{L}^{6+}$ ,

$$\begin{aligned} & E\left[\int_0^\infty f_r 1_{[0, T]}(r) dW_r \middle| \mathcal{F}_t^W\right] \\ &= \int_0^t E[f_r 1_{[0, T]}(r) | \mathcal{F}_r^W] dW_r - \int_0^t \left\{ \left( \int_0^r \bar{U}_a(r, u; f 1_{[0, T]}(\cdot)) dW_u \right) \lambda_a(r)^{-1} \right\} dM_r^a. \end{aligned}$$

*Proof.*

Note that

$$E\left[\int_0^\infty \xi 1_{(s_0, s_1]}(r) dW_r \middle| \mathcal{F}_\infty^W\right] = E[\xi | \mathcal{F}_\infty^W](W_{s_1} - W_{s_0}).$$

By Proposition 3.6, we have

$$E[\xi | \mathcal{F}_\infty^W] = E[\xi | \mathcal{F}_{s_0}^W] - \int_{s_0}^\infty \bar{U}_a(r, s_0; \xi 1_{[s_0, \infty)}(\cdot)) \lambda_a(r)^{-1} dM_r^a$$

and then

$$\begin{aligned} & E\left[\int_0^\infty \xi 1_{(s_0, s_1]}(r) dW_r \middle| \mathcal{F}_\infty^W\right] \\ &= \int_{s_0}^{s_1} E[\xi | \mathcal{F}_r^W] dW_r - \int_{s_0}^\infty \left\{ \bar{U}_a(r, s_0; \xi 1_{[s_0, s_1]}(\cdot))(W_{r \wedge s_1} - W_{s_0}) \lambda_a(r)^{-1} \right\} dM_r^a \\ &= \int_0^\infty E[\xi 1_{(s_0, s_1]}(r) | \mathcal{F}_r^W] dW_r - \int_{s_0}^\infty \left( \int_0^{r \wedge s_1} \left\{ \bar{U}_a(r, s_0; \xi 1_{(s_0, s_1]}(\cdot)) dW_u \right\} \lambda_a(r)^{-1} \right) dM_r^a. \end{aligned}$$

Here we note that

$$\begin{aligned} & 1_{\{\tau^a > s_0\}} \left( \frac{\partial g}{\partial x}(t - s \wedge \tau^a, B_{s \wedge \tau^a}^a) - \frac{\partial g}{\partial x}(t - s_0, B_{s_0}^a) \right) \\ &= 1_{\{\tau^a > s_0\}} \int_{s_0}^{s \wedge \tau^a} \frac{\partial^2 g}{\partial x^2}(t - r, B_r^a) dB_r^a, \quad s \in (s_0, t). \end{aligned}$$

Then we have

$$\begin{aligned} H_a^{(1)}(t, s_0; \xi 1_{[s_0, s_1]}(\cdot)) &= E[1_{\{\tau^a > s_0\}} \xi 1_{[s_0, s_1]}(\cdot) \left( \frac{\partial g}{\partial x}(t - s \wedge \tau^a, B_{s \wedge \tau^a}^a) \right) | \mathcal{G}_{s_0}^W] \\ &= H_a^{(1)}(t, s; \xi 1_{[s_0, s_1]}(\cdot)), \quad s \in (s_0, t \wedge s_1). \end{aligned}$$

Also we have

$$\begin{aligned} &1_{\{\tau^a > s_0\}} (\Phi(t - s \wedge \tau^a, B_{s \wedge \tau^a}^a) - \Phi(t - s_0, B_{s_0}^a)) \\ &= 1_{\{\tau^a > s_0\}} \int_{s_0}^{s \wedge \tau^a} g(t - r, B_r^a) dB_r^a, \quad s \in (s_0, t \wedge s_1). \end{aligned}$$

Thus we have

$$U_a(t, s_0; \xi 1_{[s_0, t \wedge s_1]}(\cdot)) = U_a(t, s; \xi 1_{[s_0, t \wedge s_1]}(\cdot)), \quad s \in (s_0, t \wedge s_1).$$

Since

$$\bar{U}_a(r, u; \xi 1_{(s_0, s_1]}(\cdot)) = 0, \quad r \in [0, s_0],$$

we can see that  $\int_0^\infty (\int_0^T \bar{U}_a(r, u; \xi 1_{(s_0, s_1]}(\cdot)) dW_u) \lambda_a(r)^{-1} dM_r^a$  is well defined.

Then we have the first assertion. For  $\tilde{\xi} \in \tilde{\mathcal{L}}_0$ , we have the following by the first assertion,

$$E\left[\int_0^T \tilde{\xi}_r dW_r | \mathcal{F}_\infty^W\right] = \int_0^T E[\tilde{\xi}_r | \mathcal{F}_r^W] dW_r - \int_0^T \left\{ \left( \int_0^r \bar{U}_a(r, u; \tilde{\xi}) dW_u \right) \lambda_a(r)^{-1} \right\} dM_r^a.$$

Let us take  $\{\tilde{\xi}_n\} \in \tilde{\mathcal{L}}_0$  such that

$$\lim_{n \rightarrow \infty} E\left[\int_0^T |\tilde{\xi}_n(r) - f_r| dr\right] = 0 \text{ for all } T > 0.$$

Since  $\sigma\{W_t; t \geq 0\}$  and  $\sigma\{N_t; t \geq 0\}$  are independent, we have

$$\begin{aligned} &E\left[\int_0^T \left| \int_0^r (\bar{U}_a(r, u; \tilde{\xi}_n) - \bar{U}_a(r, u; f)) dW_u \right| \lambda_a(r)^{-1} dN_r^a\right] \\ &= E\left[\int_0^T E\left[\left| \int_0^r (\bar{U}_a(r, u; \tilde{\xi}_n) - \bar{U}_a(r, u; f)) dW_u \right| \lambda_a(r)^{-1} dN_r^a\right]\right] \\ &= \int_0^T E\left[\left| \int_0^r (\bar{U}_a(r, u; \tilde{\xi}_n - f) dW_u \right| q_a(r) dr\right] \\ &\leq \int_0^T E\left[\int_0^r (\bar{U}_a(r, u; \tilde{\xi}_n - f)^2 du\right]^{1/2} dr \\ &\rightarrow 0, \quad \text{as } n \rightarrow \infty, \text{ for all } T > 0 \end{aligned}$$

by Proposition 2.3 for  $f \in \mathcal{L}^{6+}$ . So we have Assertion. ■

Fourth, we show Theorem 1.1(4) as follows.

**Proposition 3.9** *Let  $T, t > 0$  and  $\hat{f}^j \in \mathcal{L}^{2+}, j = 1, \dots, d$ . Then we have*

$$E\left[\int_0^t \hat{f}_s^j d\hat{B}_s^j | \mathcal{F}_t^W\right] = 0, j = 1, \dots, d.$$

*Proof.* Because  $B, \hat{B}$  and  $W$  are independent and  $\mathcal{F}_t^W \subset \sigma\{B_s, W_s; s \in [0, \infty)\}$ ,

$$E\left[\sum_{j=1}^d \int_0^t \hat{f}_s^j d\hat{B}_s^j | \mathcal{F}_t^W\right] = E\left[\sum_{j=1}^d \int_0^t \hat{f}_s^j d\hat{B}_s^j\right] = 0.$$

■

Finally, we state Nakagawa's [5] representation theorem using a different expression.

**Proposition 3.10** *Let  $f \in \mathcal{L}^{4+}$ . Then we have*

$$\hat{H}_a^{(2)}(t; f) = \lim_{u \uparrow t} E\left[\left(\int_0^u 1_{\{\tau^a > s\}} f_s dB_s^a\right) \frac{\partial g}{\partial x}(t - u, B_u^a) | \mathcal{G}_t^W\right].$$

*The right-hand side of the above corresponds to the representation theorem given by Nakagawa [5].*

*Proof.* Note that

$$\left(-\frac{\partial}{\partial u} + \frac{1}{2} \frac{\partial^2}{\partial x^2}\right) \frac{\partial g}{\partial x}(t - u, x) = 0$$

and so

$$d_u \frac{\partial g}{\partial x}(t - u, B_u^a) = \frac{\partial^2 g}{\partial x^2}(t - u, B_u^a) dB_u^a, \quad u < t.$$

By Ito formula, we have

$$\begin{aligned} & d_u \left( \left( \int_0^u 1_{\{\tau^a > s\}} f_s dB_s^a \right) \frac{\partial g}{\partial x}(t - u, B_u^a) \right) \\ &= 1_{\{\tau^a > u\}} f_u \frac{\partial g}{\partial x}(t - u, B_u^a) dB_u^a + \left( \int_0^u 1_{\{\tau^a > s\}} f_s dB_s^a \right) \frac{\partial^2 g}{\partial x^2}(t - u, B_u^a) dB_u^a \\ & \quad + 1_{\{\tau^a > u\}} f_u \frac{\partial^2 g}{\partial x^2}(t - u, B_u^a) du, \quad u < t. \end{aligned}$$



And then

$$\begin{aligned}
& E[(\int_0^u 1_{\{\tau^a > s\}} f_s dB_s^a) \frac{\partial g}{\partial x}(t-u, B_u^a) | \mathcal{G}_t^W] \\
&= \int_0^u E[1_{\{\tau^a > s\}} f_s \frac{\partial^2 g}{\partial x^2}(t-s, B_s^a) | \mathcal{G}_t^W] ds \\
&= \int_0^u E[1_{\{\tau^a > s\}} f_s \frac{\partial^2 g}{\partial x^2}(t-s, B_s^a) | \mathcal{G}_s^W] ds \\
&= \int_0^u H^{(2)}(t, s; f) ds, \quad u < t.
\end{aligned}$$

So we have

$$\hat{H}_a^{(2)}(t; f) = \lim_{u \uparrow t} E[(\int_0^u 1_{\{\tau^a > s\}} f_s dB_s^a) \frac{\partial g}{\partial x}(t-u, B_u^a) | \mathcal{G}_t^W].$$

■

## 4 Equivalent probability measures

We now state a representation theorem for a filtering model with first-passage-type stopping time. Note that  $I$ ,  $F$  are defined in Equations (9) and (10). Operators  $\tilde{D}_0, \tilde{D}_1, \tilde{D}_2$  and  $\tilde{L}$  are defined in Equations (11). As we defined in Equation (8), let

$$\rho_t = 1 + \int_{0+}^t \rho_{s-} (b_0(s, X_s, Z_s) d\tilde{B}_s + \beta(s, X_{s \wedge \tau}, Y_s) d\tilde{W}_s).$$

Let  $F \in \Sigma$  be given by Equation (10) in the Introduction. Then we have

$$\begin{aligned}
F_t &= F_0 + \int_0^t (f_1(s) - \beta(s, X_s, Y_s) f_2(s) - b_0(s, X_s, Z_s) f_3(s)) ds \\
&\quad + \int_0^t f_2(s) d\tilde{W}_s + \int_0^t f_3(s) d\tilde{B}_s + \int_0^t f_4(s) d\hat{B}_s
\end{aligned}$$

and so

$$\begin{aligned}
\rho_t F_{t \wedge \tau} &= \rho_0 F_0 + \int_0^t F_{s \wedge \tau} d\rho_s + \int_0^{t \wedge \tau} \rho_{s-} dF_s + [\rho, F]_{t \wedge \tau} \\
&= F_0 + \int_{0+}^t \rho_{s-} (\tilde{D}_1 F)_s d\tilde{B}_s + \int_{0+}^t \rho_{s-} (\tilde{D}_2 F)_s d\hat{B}_s \\
&\quad + \int_{0+}^t \rho_{s-} (\tilde{D}_0 F)_s d\tilde{W}_s + \int_{0+}^t \rho_{s-} (\tilde{L} F)_s ds.
\end{aligned}$$

Let

$$V(t, s; f) = \tilde{E}[\rho_{s-} 1_{\{\tau > s\}} f_s (2\Phi(t-s, X_s) - 1) | \mathcal{G}_s^Y], \quad (21)$$

$$\bar{V}(t, s; f) = e^{\int_0^t \lambda_{x_0}(r) dr} (I^{(1)}(t, s; f) + \lambda_{x_0}(t) V(t, s; f)), \quad (22)$$

$$\bar{I}(t, s; f) = e^{\int_0^t \lambda_{x_0}(r) dr} \left( \int_0^s I^{(2)}(t, u; f) du + 2\lambda_{x_0}(t) \int_0^s I^{(0)}(t, u; f) du \right), \quad (23)$$

$$\hat{V}(r, s; F) = \tilde{\rho}_{r-}^{-1} e^{\int_0^r \lambda_{x_0}(u) du} (\hat{V}_1(r, s; F) + \lambda_{x_0}(r) \hat{V}_2(r, s; F)), \quad s \leq r \quad (24)$$

where

$$\hat{V}_2(r, s; F) = \int_0^s V(r, u; \widetilde{D}_0 F) d\widetilde{W}_u + \int_0^s \left( V(r, u; \widetilde{L} F) + 2I^{(0)}(r, u; \widetilde{D}_1 F) \right) du$$

for  $f \in \mathcal{L}^{6+}$  and  $F \in \Sigma$ . Then we have the following by Theorem 1.1.

$$\begin{aligned} & \tilde{E}[\rho_t F_{t \wedge \tau} | \mathcal{F}_t] = F_0 \quad (25) \\ & - \int_0^{t \wedge \tau} \left\{ \left( \bar{I}(r, r; \widetilde{D}_1 F) + \left( \int_0^r \bar{V}(r, u; \widetilde{D}_0 F) d\widetilde{W}_u \right) + \left( \int_0^r \bar{V}(r, u; \widetilde{L} F) du \right) \lambda_{x_0}(r)^{-1} \right\} d\widetilde{M}_r \right. \\ & \quad + \int_0^{t \wedge \tau} \tilde{E}[\rho_{r-} (\widetilde{L} F)_r | \mathcal{F}_r] dr + \int_0^{t \wedge \tau} \tilde{E}[\rho_{r-} (\widetilde{D}_1 F)_r | \mathcal{F}_r] d\widetilde{W}_r \\ & \quad = F_0 - \int_0^t \tilde{\rho}_{r-} \hat{V}(r, r; F) \lambda_{x_0}(r)^{-1} d\widetilde{M}_r \\ & \quad \left. + \int_0^t \tilde{E}[\rho_{r-} (\widetilde{L} F)_r | \mathcal{F}_r] dr + \int_0^t \tilde{E}[\rho_{r-} (\widetilde{D}_1 F)_r | \mathcal{F}_r] d\widetilde{W}_r. \right\} \end{aligned}$$

Here we note that

$$\begin{aligned} & \bar{I}(r; \rho(\widetilde{D}_1 F)) + \int_0^r \bar{V}(r, u; \widetilde{D}_0 F) d\widetilde{W}_u + \int_0^r \bar{V}(r, u; \widetilde{L} F) du \\ & = e^{\int_0^r \lambda_{x_0}(u) du} \left\{ \left( \int_0^r I^{(1)}(r, u; \widetilde{D}_0 F) d\widetilde{W}_u + \int_0^r (I^{(2)}(r, u; \widetilde{D}_1 F) + I^{(1)}(r, u; \widetilde{L} F)) du \right) \right. \\ & \quad \left. + \lambda_{x_0}(r) \left( \int_0^r V(r, u; \widetilde{D}_0 F) d\widetilde{W}_u + \int_0^r (V(r, u; \widetilde{L} F) + 2 \int_0^r I^{(0)}(r, u; \widetilde{D}_1 F)) du \right) \right\} \\ & = \tilde{\rho}_{r-} \hat{V}(r, r; F). \end{aligned}$$

We will show that  $\hat{V}(r, r; F) = \hat{V}(r; F)$  and that these can be written without using stochastic integrals by Propositions 4.1 and 4.2.

**Proposition 4.1** *Let  $T > 0$  and  $F \in \Sigma$ . Then we have*

$$\hat{V}_1(r, s; F) = -\frac{\partial g}{\partial x}(r, x_0)F_0 + I^{(1)}(r, s; F), \quad 0 < s < r \leq T$$

and we can see that the right-hand side of the above equation can be defined even at  $r = s$  by  $r \downarrow s$ . Note that  $\hat{V}_1$  is defined in Equation (12).

*Proof.* Because  $\frac{\partial g}{\partial r}(r, x) - \frac{1}{2}\frac{\partial^2 g}{\partial x^2}(r, x) = 0$ , we have

$$\frac{\partial g}{\partial x}(r - s, X_s) = \int_0^s \frac{\partial^2 g}{\partial x^2}(r - u, X_u) d\tilde{B}_u.$$

So we have

$$\begin{aligned} & d\left(\frac{\partial g}{\partial x}(r - u, X_u)\rho_u F_u\right) \\ &= \rho_u\left(\left(\frac{\partial^2 g}{\partial x^2}(r - u, X_u)F_s + \frac{\partial g}{\partial x}(r - u, X_u)(\tilde{D}_1 F)_u\right)d\tilde{B}_u\right. \\ &+ \frac{\partial g}{\partial x}(r - u, X_u)\rho_u(\tilde{D}_2 F)_u d\hat{B}_u + \frac{\partial g}{\partial x}(r - u, X_u)\rho_u(\tilde{D}_0 F)_u d\tilde{W}_u \\ &+ \left.\left(\frac{\partial g}{\partial x}(r - u, X_u)(\tilde{L} F)_u + \frac{\partial^2 g}{\partial x^2}(r - u, X_u)\rho_u(\tilde{D}_1 F)_u\right)du\right). \end{aligned}$$

Since  $\mathcal{G}_s^Y$ ,  $\sigma\{\tilde{B}_u, \hat{B}_u; u \leq s\}$  and  $\sigma\{\tilde{M}_u; u \leq s\}$  are independent, we have the following for  $r > s$ .

$$\begin{aligned} & \tilde{E}\left[\frac{\partial g}{\partial x}(r - s, X_s)1_{\{\tau > s\}}\rho_s F_s | \mathcal{G}_s^Y\right] \\ &= \frac{\partial g}{\partial x}(r, x_0)F_0 + \int_0^s I^{(1)}(r, u; \tilde{D}_0 F) d\tilde{W}_u + \int_0^s (I^{(2)}(r, u; \tilde{D}_1 F) + I^{(1)}(r, u; \tilde{L} F)) du \\ &= \frac{\partial g}{\partial x}(r, x_0)F_0 + \hat{V}_1(r, s; F). \end{aligned}$$

Then we have our assertion. ■

**Proposition 4.2** *Let  $T > 0$  and  $F \in \Sigma$ . Then we have*

$$\hat{V}_2(r, s; F) = -(2\Phi(r, x_0) - 1)F_0 + V(r, s; F), \quad 0 < s < r \leq T$$

and

$$\lim_{s \uparrow r} \hat{V}_2(r, s; F) = -(2\Phi(r, x_0) - 1)F_0 + \tilde{E}[1_{\{\tau > r\}}\rho_r F_r | \mathcal{G}_r^Y].$$

In particular,  $\hat{V}(r, r; F) = \hat{V}(r; F)$ . Note that  $\hat{V}(r; F)$  and  $\hat{V}(r, s; F)$  are defined in Equation (12) and (24), respectively.

*Proof.* Because  $\frac{\partial \Phi}{\partial r}(r, x) - \frac{1}{2} \frac{\partial^2 \Phi}{\partial x^2}(r, x) = 0$  and  $\frac{\partial \Phi}{\partial x}(r, x) = g(r, x)$ ,

$$\Phi(r - s, X_s) = \int_0^s g(r - u, X_u) d\tilde{B}_u.$$

So we have

$$\begin{aligned} & (2\Phi(r - s, X_s) - 1)\rho_s F_s \\ &= (2\Phi(r, x_0) - 1)F_0 + 2 \int_0^s \rho_{u-} F_u g(r - u, X_u) d\tilde{B}_u \\ &+ \int_0^s \{ (2\Phi(r - u, X_u) - 1)(\rho_{u-}(\tilde{D}_1 F)_u d\tilde{B}_u + \rho_{u-}(\tilde{D}_2 F)_u d\hat{B}_u + \rho_{u-}(\tilde{D}_0 F)_u d\tilde{W}_u + \rho_{u-}(\tilde{L} F)_u du) \} \\ &+ 2 \int_0^s \rho_{u-} g(r - u, X_u) (\tilde{D}_1 F)_u du. \end{aligned}$$

And then we have the following by Lemma 3.1, which gives the first assertion.

$$\begin{aligned} & V(r, s; F) \\ &= \tilde{E}[(2\Phi(r, x_0) - 1)F_0 + \int_0^s 1_{\{\tau > u\}}(2\Phi(r - u, X_u) - 1)\rho_{u-}(\tilde{D}_0 F)_u d\tilde{W}_u \\ &+ \int_0^s 1_{\{\tau > u\}}((2\Phi(r - u, X_u) - 1)\rho_{u-}(\tilde{L} F)_u + 2g(r - u, X_u)\rho_{u-}(\tilde{D}_1 F)_u) du | \mathcal{G}_r^Y] \\ &= (2\Phi(r, x_0) - 1)F_0 + \int_0^s V(r, u; \tilde{D}_0 F) d\tilde{W}_u + \int_0^s (V(r, u; \tilde{L} F) + 2I^{(0)}(r, u, \tilde{D}_1 F)) du \\ &= (2\Phi(r, x_0) - 1)F_0 + \hat{V}_2(r, s; F). \end{aligned}$$

Then we have

$$\begin{aligned} & (2\Phi(r, x_0) - 1)F_0 + \lim_{s \uparrow r} \hat{V}_2(r, s; F) \\ &= \lim_{s \uparrow r} V(r, s; F) \\ &= \tilde{E}[(2\Phi(r - r \wedge \tau, X_{r \wedge \tau}) - 1)\rho_{r \wedge \tau} F_{r \wedge \tau} | \mathcal{G}_r^Y] \\ &= \tilde{E}[1_{\{\tau > r\}}(2\Phi(0, X_r) - 1)\rho_r F_r | \mathcal{G}_r^Y] - \tilde{E}[1_{\{\tau \leq r\}}(2\Phi(r - \tau, 0) - 1)\rho_\tau F_\tau | \mathcal{G}_r^Y] \\ &= \tilde{E}[1_{\{\tau > r\}}\rho_r F_r | \mathcal{G}_r^Y]. \end{aligned}$$

So we can see that

$$\begin{aligned} & \hat{V}(r, r; F) \\ &= \tilde{\rho}_{r-}^{-1} e^{\int_0^r \lambda_{x_0}(u) du} (\hat{V}_1(r, r; F) + \lambda_{x_0}(r) V(r, r; F)) \\ &= \tilde{\rho}_{r-}^{-1} e^{\int_0^r \lambda_{x_0}(u) du} \left\{ \hat{V}_1(r, r; F) + \lambda_{x_0}(r) \left( -(2\Phi(r, x_0) - 1)F_0 + \tilde{E}[1_{\{\tau > r\}}\rho_r F_r | \mathcal{G}_r^Y] \right) \right\} \\ &= \hat{V}(r; f) \end{aligned}$$

by the first assertion of this Proposition and Proposition 4.1.  $\blacksquare$

We now state Proposition 4.3, Lemma 4.4 and Proposition 4.5 for Theorem 1.2(1).

**Proposition 4.3** *Let  $\xi$  be a  $\mathcal{B}$ -measurable process. Then, we have*

$$E[\xi_{t \wedge \tau} | \mathcal{H}] = \frac{\tilde{E}[\rho_t \xi_{t \wedge \tau} | \mathcal{H}]}{\tilde{E}[\rho_t | \mathcal{H}]}, \quad \mathcal{H} \subset \mathcal{B}_t.$$

*Proof.* For  $A \in \mathcal{H} \subset \mathcal{B}_t$ , we have

$$\begin{aligned} E[\xi_{t \wedge \tau}, A] &= E[E[\xi_{t \wedge \tau} | \mathcal{H}], A] = \tilde{E}[\rho_T E[\xi_{t \wedge \tau} | \mathcal{H}], A] \\ &= \tilde{E}[\tilde{E}[\rho_T | \mathcal{B}_t] E[\xi_{t \wedge \tau} | \mathcal{H}], A] = \tilde{E}[\tilde{E}[\rho_t | \mathcal{H}] E[\xi_{t \wedge \tau} | \mathcal{H}], A]. \end{aligned}$$

At the same time,

$$E[\xi_{t \wedge \tau}, A] = \tilde{E}[\rho_T \xi_{t \wedge \tau}, A] = \tilde{E}[\tilde{E}[\rho_T | \mathcal{B}_t] \xi_{t \wedge \tau}, A] = \tilde{E}[\rho_t \xi_{t \wedge \tau}, A] = \tilde{E}[\tilde{E}[\rho_t \xi_{t \wedge \tau} | \mathcal{H}], A].$$

$\blacksquare$

**Lemma 4.4**

$$\begin{aligned} \tilde{E}[\rho_t F_{t \wedge \tau} | \mathcal{F}_t] &= F_0 - \int_0^{t \wedge \tau} \tilde{\rho}_{r-} \hat{V}(r; F) \lambda_{x_0}(r)^{-1} d\tilde{M}_r \\ &\quad + \int_0^{t \wedge \tau} \tilde{E}[\rho_{r-} (\tilde{L}F)_r | \mathcal{F}_r] dr + \int_0^{t \wedge \tau} \tilde{E}[\rho_{r-} (\tilde{D}_0 F)_r | \mathcal{F}_r] d\tilde{W}_r. \end{aligned}$$

*Proof.* Since  $\hat{V}(r, r; F) = \hat{V}(r; F)$  by Proposition 4.2, we have our assertion by Equation (25).  $\blacksquare$

Let  $\tilde{\rho}_t = \tilde{E}[\rho_t | \mathcal{F}_t]$ .

**Proposition 4.5**

$$\tilde{\rho}_t = 1 - \int_0^t \tilde{\rho}_{r-} \hat{V}(r; 1) \lambda_{x_0}(r)^{-1} d\tilde{M}_r + \int_0^t \tilde{\rho}_{r-} \tilde{E}[\beta(r, X_r, Y_r) | \mathcal{F}_r] d\tilde{W}_r$$

and

$$\begin{aligned} \tilde{\rho}_t^{-1} &= 1 - \int_0^t \tilde{\rho}_{r-}^{-1} \frac{\hat{V}(r; 1)}{\lambda_{x_0}(r) + \hat{V}(r; 1)} d\tilde{M}_r \\ &\quad + \int_0^t \tilde{\rho}_{r-}^{-1} (\tilde{E}[\beta(r, X_r, Y_r) | \mathcal{F}_r])^2 + \frac{\hat{V}(r; 1)^2}{\lambda_{x_0}(r) + \hat{V}(r; 1)} 1_{\{\tau > r\}} dr - \int_0^t \tilde{\rho}_{r-}^{-1} \tilde{E}[\beta(r, X_r, Y_r) | \mathcal{F}_r] d\tilde{W}_r. \end{aligned}$$

*Proof.* Letting  $F_t = 1$  in Lemma 4.4, we have the first assertion. Then we have

$$\begin{aligned}
\tilde{\rho}_t^{-1} &= 1 - \int_0^t \tilde{\rho}_{r-}^{-2} d\tilde{\rho}_r + \int_0^t \tilde{\rho}_{r-}^{-3} d[\tilde{\rho}, \tilde{\rho}]_r^c + \sum_{0 < r \leq t} (\tilde{\rho}_r^{-1} - \tilde{\rho}_{r-}^{-1} + \tilde{\rho}_{r-}^{-2}(\tilde{\rho}_r - \tilde{\rho}_{r-})) \\
&= 1 - \int_0^t \tilde{\rho}_{r-}^{-1} \hat{V}(r; 1) \lambda_{x_0}(r)^{-1} d\tilde{M}_r - \int_0^t \tilde{\rho}_{r-}^{-1} \tilde{E}[\beta(r, X_r, Y_r) | \mathcal{F}_r] d\tilde{W}_r \\
&\quad + \int_0^t \tilde{\rho}_{r-}^{-1} \tilde{E}[\beta(r, X_r, Y_r) | \mathcal{F}_r]^2 dr + \int_0^t \tilde{\rho}_{r-}^{-1} \frac{\hat{V}(r; 1)^2}{\lambda_{x_0}(r) + \hat{V}(r; 1)} \lambda_{x_0}(r)^{-1} dN_r.
\end{aligned}$$

Here we use the fact that

$$\begin{aligned}
&\sum_{0 < r \leq t} (\tilde{\rho}_r^{-1} - \tilde{\rho}_{r-}^{-1} + \tilde{\rho}_{r-}^{-2}(\tilde{\rho}_r - \tilde{\rho}_{r-})) \\
&= \sum_{0 < r \leq t} \frac{(\tilde{\rho}_r - \tilde{\rho}_{r-})^2}{\tilde{\rho}_{r-}^2 \tilde{\rho}_r} = \int_0^t \tilde{\rho}_{r-}^{-1} \frac{\hat{V}(r; 1)^2}{\lambda_{x_0}(r) + \hat{V}(r; 1)} \lambda_{x_0}(r)^{-1} dN_r.
\end{aligned}$$

Then we have the assertion. ■

We give Propositions 4.6, 4.7, and 4.8 for Theorem 1.2(2).

#### Proposition 4.6

$$\lambda_a(t)(2\Phi(t, a) - 1) + \frac{\partial g}{\partial x}(t, a) = 0.$$

In particular,

$$\begin{aligned}
&\hat{V}(r; F) = \\
&\tilde{\rho}_{r-}^{-1} e^{\int_0^r \lambda_{x_0}(u) du} \left( \hat{V}_1(r, r; F) + \frac{\partial g}{\partial x}(r, x_0) F_0 + \tilde{E}[1_{\{\tau > r\}} \rho_r F_r | \mathcal{G}_r^Y] \right).
\end{aligned}$$

*Proof.* Because of the well-known reflection principle of Brownian motion, we have  $q_a(t) = P[\tau^a > t] = 1 - P[\tau^a < t] = 1 - \frac{2}{\sqrt{2\pi t}} \int_0^\infty e^{-\frac{x^2}{2t}} dx = 2\Phi(t, a) - 1$ . So we have

$$\frac{\partial}{\partial t} q_a(t) = 2 \frac{\partial \Phi}{\partial x}(t, x) = 2 \int_{-\infty}^x \frac{\partial g}{\partial t}(t, y) dy = \frac{\partial g}{\partial x}(t, x).$$

Since  $\lambda_a(t) = -\frac{d}{dt} \log q_a(t)$ , we have the assertion. ■

**Proposition 4.7** *Let  $Z$  be a random variable and  $r > 0$ . Then we have*

$$\tilde{E}[Z 1_{\{\tau > r\}} | \mathcal{G}_r^Y] 1_{\{\tau > r\}} = e^{-\int_0^r \lambda_{x_0}(u) du} \tilde{E}[Z | \mathcal{F}_r] 1_{\{\tau > r\}}.$$

*Proof.* Let  $A \in \mathcal{F}_r$ . Then there exists  $B \in \mathcal{G}_r^Y$  such that  $A \cap \{\tau > r\} = B \cap \{\tau > r\}$ . Since  $\mathcal{G}_r^Y$  and  $1_{\{\tau > r\}}$  are independent, we have the following.

$$\begin{aligned}
& \tilde{E}[\tilde{E}[Z1_{\{\tau > r\}}|\mathcal{G}_r^Y]1_{\{\tau > r\}}, A] \\
&= \tilde{E}[\tilde{E}[Z1_{\{\tau > r\}}|\mathcal{G}_r^Y]1_{\{\tau > r\}}1_B] = \tilde{E}[\tilde{E}[Z1_{\{\tau > r\}}1_B|\mathcal{G}_r^Y]1_{\{\tau > r\}}] \\
&= \tilde{E}[\tilde{E}[Z1_B|\mathcal{G}_r^Y]\tilde{E}[1_{\{\tau > r\}}]] = \tilde{E}[Z, A]\tilde{P}[\tau > r] \\
&= \tilde{E}[e^{-\int_0^r \lambda_{x_0}(u)du} Z, A] = \tilde{E}[e^{-\int_0^r \lambda_{x_0}(u)du} \tilde{E}[Z|\mathcal{F}_r], A].
\end{aligned}$$

Then we have Assertion. ■

**Proposition 4.8** *Let  $F \in \Sigma$ . Assume that there exist  $C > 0$  and  $\alpha \in (0, 1)$  such that  $1_{\{|X_r| \leq 1\}}1_{\{\tau > r\}}|F_r| \leq C|X_r|^\alpha$  for  $r > 0$ . Then we have  $\hat{V}_1(r, r; F) = -\frac{\partial g}{\partial x}(r, x_0)F_0$  and*

$$\hat{V}(r; F) = \tilde{\rho}_{r-}^{-1} e^{\int_0^r \lambda_{x_0}(u)du} \lambda_{x_0}(r) \tilde{E}[1_{\{\tau > r\}} \rho_r F_r | \mathcal{G}_r^Y].$$

In particular,

$$1_{\{\tau > r\}} \hat{V}(r; F) = 1_{\{\tau > r\}} \tilde{\rho}_{r-}^{-1} \lambda_{x_0}(r) \tilde{E}[1_{\{\tau > r\}} \rho_r F_r | \mathcal{F}_r].$$

*Proof.* Let  $1 < p < \frac{2}{2-\alpha}$ ,  $q = \frac{p}{p-1}$  and  $r > s > 0$ . Then we have

$$\tilde{E}[|I^{(1)}(r, s; \tilde{F})|] \leq \tilde{E}[|\frac{\partial g}{\partial x}(r-s, X_s)| 1_{\{\tau > s\}} \rho_s |F_s|] \leq \tilde{E}[1_{\{\tau > r\}} |\frac{\partial g}{\partial x}(r-s, X_s)|^p |F_s|^p]^{\frac{1}{p}} \tilde{E}[\rho_s^q]^{\frac{1}{q}}.$$

Note that  $x_0 > 0$ . We have the following by Mean-Value Theorem.

$$e^{-\frac{(x-x_0)^2}{2s}} - e^{-\frac{(x+x_0)^2}{2s}} \leq \frac{x_0(x+x_0)}{s^2}, \quad x \in (0, \infty), \quad s \in (0, r).$$

Since  $\frac{2-p(2-\alpha)}{2} > 0$ , we have

$$\begin{aligned}
& \tilde{E}[1_{\{|X_s| \leq 1\}} 1_{\{\tau > r\}} |\frac{\partial g}{\partial x}(r-s, X_s)|^p |F_s|^p] \\
& \leq \int_0^\infty \left( \frac{1}{\sqrt{2\pi(r-s)}} \frac{x}{r-s} e^{-\frac{x^2}{2(r-s)}} \right)^p (Cx^\alpha)^p \frac{e^{-\frac{(x-x_0)^2}{2s}} - e^{-\frac{(x+x_0)^2}{2s}}}{\sqrt{2\pi s}} dx \\
& \leq x_0 C^p (r-s)^{-\frac{3p}{2}} s^{-\frac{5}{2}} \int_0^\infty x^{(1+\alpha)p} (x+x_0) e^{-\frac{px^2}{2(r-s)}} dx \\
& = x_0 C^p (r-s)^{\frac{2-p(2-\alpha)}{2}} s^{-\frac{5}{2}} \int_0^\infty y^{(1+\alpha)p} y e^{-\frac{py^2}{2}} dy \\
& + x_0^2 C^p (r-s)^{\frac{2-p(2-\alpha)}{2}} s^{-\frac{5}{2}} \int_0^\infty y^{(1+\alpha)p} e^{-\frac{py^2}{2}} dy \\
& \rightarrow 0 \quad \text{as } s \uparrow r.
\end{aligned}$$

Let  $p' > 1$ ,  $q' = \frac{p'}{p'-1}$ . For  $r > s > 0$ , we have

$$\begin{aligned}
& \widetilde{E}[1_{\{|X_s|>1\}} 1_{\{\tau>r\}} \left| \frac{\partial g}{\partial x}(r-s, X_s) \right|^p |F_s|^p] \\
& \leq \widetilde{E}[1_{\{|X_s|>1\}} 1_{\{\tau>r\}} \left| \frac{\partial g}{\partial x}(r-s, X_s) \right|^{pp'}]^{\frac{1}{p'}} E[|F_s|^{pq'}]^{\frac{1}{q'}} \\
& = \left\{ \int_1^\infty \left( \frac{1}{\sqrt{2\pi(r-s)}} \frac{x}{r-s} e^{-\frac{x^2}{2(r-s)}} \right)^{pp'} \frac{e^{-\frac{(x-x_0)^2}{2s}} - e^{-\frac{(x+x_0)^2}{2s}}}{\sqrt{2\pi s}} dx \right\}^{\frac{1}{p'}} E[|F_s|^{pq'}]^{\frac{1}{q'}} \\
& \rightarrow 0 \quad \text{as } s \uparrow r.
\end{aligned}$$

Then we have  $\hat{V}_1(r, r; F) = -\frac{\partial g}{\partial x}(r, x_0)F_0 + \lim_{s \uparrow r} I^{(1)}(r, s; F) = -\frac{\partial g}{\partial x}(r, x_0)F_0$ . By Proposition 4.6, we have the first assertion. We have the second assertion by Proposition 4.7.  $\blacksquare$

We can show Theorem 1.2 (1) and (2) using the following proposition.

**Proposition 4.9**  $\widetilde{\widetilde{M}}_t$  is  $P$ - $\mathcal{F}_t$ -martingale and  $\widetilde{\widetilde{W}}_t$  is  $P$ - $\mathcal{B}_t$ -Brownian motion.

*Proof.* By Proposition 4.5,

$$d[\widetilde{\rho}^{-1}, \widetilde{M}]_t = -\widetilde{\rho}_{t-}^{-1} \frac{\hat{V}(t; 1)}{\lambda_{x_0}(t) + \hat{V}(t; 1)} dN_t$$

and then we have

$$d(\widetilde{\rho}_t^{-1} \widetilde{M}_t) = \widetilde{\rho}_{t-}^{-1} d\widetilde{M}_t + \widetilde{M}_{t-} d(\widetilde{\rho}^{-1})_t + d[\widetilde{\rho}^{-1}, \widetilde{M}]_t = \frac{\widetilde{\rho}_{t-}^{-1} \lambda_{x_0}(t)}{\lambda_{x_0}(t) + \hat{V}(t; 1)} d\widetilde{\widetilde{M}}_t + \widetilde{M}_{t-} d(\widetilde{\rho}^{-1})_t.$$

Since  $\widetilde{\rho}_t^{-1} \widetilde{M}_t$  and  $\widetilde{\rho}_t^{-1}$  are  $P$ - $\mathcal{F}_t$ -martingale, we can see  $\widetilde{\widetilde{M}}_t$  is also  $P$ - $\mathcal{F}_t$ -martingale. We can see that  $\widetilde{\widetilde{W}}_t$  is  $P$ - $\mathcal{B}_t$ -Brownian motion by the following.

$$d(\widetilde{\rho}_t^{-1} \widetilde{W}_t) = \widetilde{\rho}_{t-}^{-1} d\widetilde{W}_t + \widetilde{W}_{t-} d(\widetilde{\rho}^{-1})_t + d[\widetilde{\rho}^{-1}, \widetilde{W}]_t = \widetilde{\rho}_{t-}^{-1} d\widetilde{\widetilde{W}}_t + \widetilde{W}_{t-} d(\widetilde{\rho}^{-1})_t.$$

Now let us prove Theorem 1.2. Let  $\hat{F}_t = \widetilde{E}[\rho_t F_{t \wedge \tau} | \mathcal{F}_t]$ , then we have the following by Lemma 4.4 and Proposition 4.5.  $\blacksquare$

$$\begin{aligned}
\hat{F}_t &= F_0 + \int_0^t \hat{f}_0(r; F) d\widetilde{\widetilde{M}}_r + \int_0^t \hat{f}_1(r; F) dr + \int_0^t \hat{f}_2(r; F) d\widetilde{\widetilde{W}}_r, \\
\widetilde{\rho}_t^{-1} &= 1 + \int_0^t \widetilde{\rho}_{r-}^{-1} \widetilde{f}_0(r) d\widetilde{\widetilde{M}}_r + \int_0^t \widetilde{\rho}_{r-}^{-1} \widetilde{f}_2(r) d\widetilde{\widetilde{W}}_r
\end{aligned}$$



where

$$\begin{aligned}
\hat{f}_0(r; F) &= -\frac{\hat{V}(r; F)}{\tilde{\lambda}(r) - \hat{V}(r; 1)} \tilde{\rho}_{r-}, \quad \hat{f}_2(r; F) = \tilde{E}[\rho_{r-}(\widetilde{D_0 F})_r | \mathcal{F}_r], \\
\hat{f}_1(r; F) &= \tilde{E}[\rho_{r-}(\tilde{L}F)_r | \mathcal{F}_r] \\
&\quad + 1_{\{\tau > r\}} \frac{\hat{V}(r; 1) \hat{V}(r; F)}{\tilde{\lambda}(r) - \hat{V}(r; 1)} \tilde{\rho}_{r-} + E[\beta(r, X_{r \wedge \tau}, Y_r) | \mathcal{F}_r] \tilde{E}[\rho_{r-}(\widetilde{D_0 F})_r | \mathcal{F}_r], \\
\tilde{f}_0(r) &= -\hat{V}(r; 1) \tilde{\lambda}(r)^{-1}, \quad \tilde{f}_2(r) = -E[\beta(r, X_r, Y_r) | \mathcal{F}_r].
\end{aligned}$$

Note that  $d\widetilde{M}_t = d\widetilde{M}_t + (1 - N_{t-}) \hat{V}(t; 1) dt$  and  $d\widetilde{W}_t = d\widetilde{W}_t + E[\beta(t, X_{t \wedge \tau}, Y_t) | \mathcal{F}_t] dt$ . Then  $E[F_{t \wedge \tau} | \mathcal{F}_t] = \tilde{\rho}_t^{-1} \hat{F}_t$ . Let  $\bar{F}_t = E[F_{t \wedge \tau} | \mathcal{F}_t]$  and we have the following.

$$\begin{aligned}
&\tilde{\rho}_t^{-1} \hat{F}_t \\
&= F_0 + \int_0^{t \wedge \tau} \tilde{\rho}_{r-}^{-1} d\hat{F}_r + \int_0^{t \wedge \tau} \hat{F}_{r-} d\tilde{\rho}_r^{-1} + [\hat{F}, \tilde{\rho}^{-1}]_{t \wedge \tau} \\
&= F_0 + \int_0^{t \wedge \tau} \tilde{\rho}_{r-}^{-1} \left( \hat{f}_0(r; F) + \tilde{f}_0(r) \hat{F}_{r-} \right) d\widetilde{M}_r + \int_0^{t \wedge \tau} \tilde{\rho}_{r-}^{-1} \left( \hat{f}_1(r; F) + \tilde{f}_2(r) \hat{f}_2(r; F) \right) dr \\
&\quad + \int_0^{t \wedge \tau} \tilde{\rho}_{r-}^{-1} \left( \hat{f}_2(r; F) + \tilde{f}_2(r) \hat{F}_{r-} \right) d\widetilde{W}_r + \sum_{0 < r \leq t \wedge \tau} (\tilde{\rho}_r^{-1} - \tilde{\rho}_{r-}^{-1}) (\hat{F}_r - \hat{F}_{r-}) \\
&= F_0 + \int_0^{t \wedge \tau} \tilde{\rho}_{r-}^{-1} \left( \hat{f}_0(r; F) + \tilde{f}_0(r) \hat{F}_{r-} + \tilde{f}_0(r) \hat{f}_0(r; F) \right) d\widetilde{M}_r \\
&\quad + \int_0^{t \wedge \tau} \tilde{\rho}_{r-}^{-1} \left( \hat{f}_1(r; F) + \tilde{f}_2(r) \hat{f}_2(r; F) + 1_{\{\tau > r\}} \tilde{f}_0(r) \hat{f}_0(r; F) \tilde{\lambda}(r) \right) dr \\
&\quad + \int_0^{t \wedge \tau} \tilde{\rho}_{r-}^{-1} \left( \hat{f}_2(r; F) + \tilde{f}_2(r) \hat{F}_{r-} \right) d\widetilde{W}_r.
\end{aligned}$$

Here we note that

$$\sum_{0 < r \leq t} (\tilde{\rho}_r^{-1} - \tilde{\rho}_{r-}^{-1}) (\hat{F}_r - \hat{F}_{r-}) = \int_0^t \tilde{\rho}_{r-}^{-1} \tilde{f}_0(r) \hat{f}_0(r; F) dN_r.$$

Then we have Theorem 1.2(1) as the following.

$$\begin{aligned}
E[F_{t \wedge \tau} | \mathcal{F}_t] &= F_0 - \int_0^t 1_{\{\tau > r\}} \left( \hat{V}(r; F) + \hat{V}(r; 1) \bar{F}_{r-} \right) \tilde{\lambda}(r)^{-1} d\widetilde{M}_r \\
&\quad + \int_0^t 1_{\{\tau > r\}} E[1_{\{\tau > r\}} (\tilde{L}F)_r | \mathcal{F}_r] dr \\
&\quad + \int_0^t \left( E[(\widetilde{D_0 F})_r | \mathcal{F}_r] - E[\beta(r, X_r, Y_r) | \mathcal{F}_r] \bar{F}_{r-} \right) d\widetilde{W}_r.
\end{aligned}$$

If there exist  $C > 0$  and  $\alpha \in (0, 1)$  such that  $1_{\{|X_t| \leq 1\}} 1_{\{\tau > t\}} |F_t| \leq C |X_t|^\alpha$  for  $t > 0$ , we have

$$\begin{aligned} 1_{\{\tau > r\}} \hat{V}(r; F) &= 1_{\{\tau > r\}} \tilde{\rho}_{r-}^{-1} \lambda_{x_0}(r) \tilde{E}[1_{\{\tau > r\}} \rho_r F_r | \mathcal{F}_r] \\ &= 1_{\{\tau > r\}} \tilde{\rho}_{r-}^{-1} \lambda_{x_0}(r) e^{\int_0^r \lambda_{x_0}(u) du} \tilde{E}[1_{\{\tau > r\}} \rho_r F_r | \mathcal{G}_r] \end{aligned}$$

by Proposition 4.7 and Proposition 4.8. Then we have

$$\begin{aligned} \tilde{f}_0(r; F) &= -1_{\{\tau > r\}} \left( \hat{V}(r; F) + \hat{V}(r; 1) \bar{F}_{r-} \right) \tilde{\lambda}(r)^{-1} \\ &= -1_{\{\tau > r\}} \frac{\lambda_{x_0}(r) \tilde{\rho}_{r-}^{-1} e^{\int_0^r \lambda_{x_0}(u) du} \tilde{E}[1_{\{\tau > r\}} \rho_r F_r | \mathcal{G}_r] + \hat{V}(r; 1) \bar{F}_{r-}}{\lambda_{x_0}(r) + \hat{V}(r; 1)} \\ &= -1_{\{\tau > r\}} \bar{F}_{r-}, \end{aligned}$$

which gives Theorem 1.2(2).

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